●●●●○ Level 4 · Mathematics M4.1

Powers, roots, surds and standard form

Use the laws of powers, work with roots and surds, and write huge and tiny numbers in standard form.

Learn the laws of powers, then zero, negative and fractional powers. Write very big and very small numbers in standard form, tell rational numbers from irrational ones, simplify surds and find the bounds of a rounded number.

YOU ARE READY FOR THIS BOOK IF

You can work out 2^5 and the square root of 144.

This book	**M4.1 Powers, roots, surds and standard form**
One step down	M3.1 Negative numbers, factors and powers

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M4.1.1 Laws of powers

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M4.1.6 Rational and irrational numbers

M4.1.7 Surds

M4.1.8 Accuracy and bounds

Key words, A to Z

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M4.1.1 CHAPTER 1

Laws of powers

Why is $a^3 \times a^4$ equal to a^7, and not a^{12}?

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** $a^3 \times a^4 = a^{12}$, because $3 \times 4 = 12$.
- ☐ **B** $a^3 \times a^4 = 2a^7$, because there are two powers of a.
- ☐ **C** $a^3 \times a^4 = a^7$, because there are $3 + 4 = 7$ letters a multiplied together.

IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Negative numbers, factors and powers, M3.1.7. If a^3 or 5^2 looks new, read this first.
- Level 3, Letters for numbers and simple equations, M3.8.1. If a letter standing for a number looks new, read this first.

BY THE END YOU CAN

- Simplify $a^3 \times a^4$ and $a^7 \div a^2$
- Simplify $(a^3)^2$ and $(2a^2)^3$

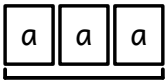
Ravi sees $a^3 \times a^4$ on a page. He asks how many a's are being multiplied together. Here a stands for any number.

In a^3, the letter a is the **base**. The small raised 3 is the **index**. It counts the copies of a that are multiplied. So $a^3 = a \times a \times a$. The plural of index is indices.

DRAW IT FIRST

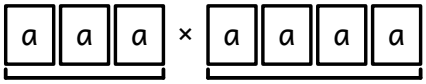
Put the a's in boxes

$a^3 \times a^4$



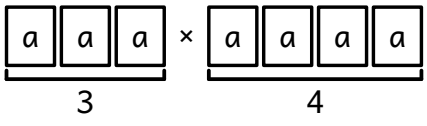
1 a^3 means three a's multiplied together. Draw them as three boxes.

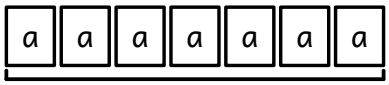
$a^3 \times a^4$



2 a^4 means four a's. Draw four more boxes after the × sign.

$a^3 \times a^4$





$= a^7$

3 Now all the boxes are multiplied. Count them: 3 + 4 = 7 boxes, so the product is a^7.

This is the first law. To multiply powers of the same base, add the indices. It only works when the base is the same.

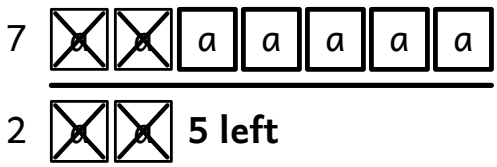
WORKED EXAMPLE

Simplify $a^3 \times a^4$.

1. a^3 is 3 a's multiplied together.
2. a^4 is 4 a's. Together there are $3 + 4 = 7$ a's.
3. Seven a's multiplied together is a^7.

Answer: $a^3 \times a^4 = a^7$

Now divide. $a^7 \div a^2$ is a fraction with seven a's above and two a's below. A letter divided by itself is 1, because $a \div a = 1$ whenever a is not 0. So each a below cancels one a above.



$a^7 \div a^2$: two pairs cross out, and 5 a's are left.

This is the second law. To divide powers of the same base, subtract the indices: the index on top minus the index below.

WORKED EXAMPLE

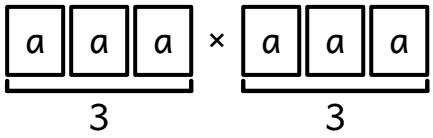
Simplify $a^7 \div a^2$.

1. Write it as a fraction. There are 7 a's above and 2 a's below.
2. Each a below cancels one a above, because $a \div a = 1$.
3. 2 pairs cancel, leaving $7 - 2 = 5$ a's above.
4. So the answer is a^5.

Answer: $a^7 \div a^2 = a^5$

A power can be raised to a power. $(a^3)^2$ means two copies of a^3 multiplied together.

$$(a^3)^2 = a^3 \times a^3$$



$(a^3)^2$ is 2 groups of 3 a's: $2 \times 3 = 6$ a's.

This is the third law. To raise a power to a power, multiply the indices. When the inside of the bracket is a product, such as $2 \times a^2$, the index applies to every part of it. This does not work for a sum: $(a + b)^2$ is not $a^2 + b^2$.

WORKED EXAMPLE

Simplify $(2a^2)^3$.

1. The index 3 means 3 copies are multiplied: $2a^2 \times 2a^2 \times 2a^2$.
2. Numbers first: $2 \times 2 \times 2 = 8$.
3. Letters next: $a^2 \times a^2 \times a^2$ has $2 + 2 + 2 = 6$ a's multiplied.
4. Together, $(2a^2)^3 = 8a^6$.

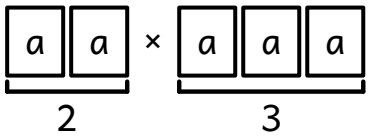
Answer: $(2a^2)^3 = 8a^6$

SEE IT THREE WAYS

With things

Write a in a box 2 times, then 3 more times. Count the boxes.

As a picture



In symbols

$a^2 \times a^3 = a \times a \times a \times a \times a = a^5$, and $2 + 3 = 5$.

Indices	Example
add	$a^3 \times a^4 = a^7$
subtract	$a^7 \div a^2 = a^5$
multiply	$(a^3)^2 = a^6$

What to do with the indices: add them to multiply, subtract them to divide, multiply them for a power of a power.

TRY IT

Test the first law with numbers

You need: paper and a pencil.

1. Work out $2^3 = 2 \times 2 \times 2$ and $2^4 = 2 \times 2 \times 2 \times 2$.
2. Multiply the two answers together.
3. Now work out 2^7 and compare.

What you will see: $2^3 = 8$ and $2^4 = 16$. Then $8 \times 16 = 128$, and $2^7 = 128$ as well. The law gives the same answer as working it out.

If you cannot try it, this is what happens.

Sum	Value
$2^3 \times 2^4 = 8 \times 16$	128
2^7	128

A WRONG IDEA MANY PEOPLE HAVE

~~**$a^3 \times a^4 = a^{12}$, because you multiply the indices.**~~

Why it feels right: The sign says times, so it feels as if everything should be multiplied.

The test that breaks it: Put $a = 2$. Then $2^3 \times 2^4 = 8 \times 16 = 128$, but $2^{12} = 4096$. The two do not match, and $2^7 = 128$ does.

What is true: When you multiply powers of the same base, you add the indices. You multiply indices only when a power is raised to a power.

A WRONG IDEA MANY PEOPLE HAVE

~~**$a^3 \times b^4 = ab^7$.**~~

Why it feels right: The indices 3 and 4 are there to be added.

The test that breaks it: Put $a = 2$ and $b = 3$. Then $a^3 \times b^4 = 8 \times 81 = 648$, but $ab^7 = 2 \times 2187 = 4374$.

What is true: The laws need the same base. $a^3 \times b^4$ has two different letters, so it stays as it is.

KEY WORDS

base
the number or letter whose copies are multiplied, such as a in a^3.

index
the small raised number. It counts the copies of the base that are multiplied.

indices
more than one index.

BACK TO YOUR GUESS

The answer is C: $a^3 \times a^4 = a^7$, because there are 3 + 4 = 7 letters a multiplied together.

Three a's multiplied by four a's make seven a's multiplied together. So the answer is a^7. You add the indices. You do not multiply them.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. Simplify $a^5 \times a^2$.

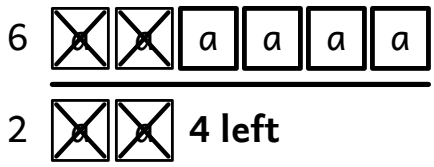
   A a^{10}

   B a^7

   C a^3

   Answer at the back of the book.

2. The picture shows $a^6 \div a^2$. Which answer is right?



   A a^3

   B a^8

   C a^4

   Answer at the back of the book.

3. Simplify $(a^3)^2$.

   A a^6

   B a^5

   C a^9

   Answer at the back of the book.

4. Simplify $(3a^4)^2$.

   A $3a^8$

   B $6a^8$

   C $9a^6$

   D $9a^8$

   Answer at the back of the book.

5. Can $a^3 \times b^4$ be written as one power of a letter by adding the indices?

   A Yes. The answer is ab^7.

   B Yes. The answer is $(ab)^{12}$.

   C No. The letters are different, so the indices cannot be added.

   Answer at the back of the book.

6. Simplify $a^8 \div a^3$. What is the index in the answer?

   Answer at the back of the book.

7. Simplify $a^4 \times a^5 \div a^3$. What is the index in the answer?

   Answer at the back of the book.

8. Work out $2^6 \div 2^4$ as a power of 2, and then as a number. What is the number?

   Answer at the back of the book.

9. Simplify $(a^4)^3 \times a^2$. What is the index in the answer?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Same base: multiply, add the indices. Divide, subtract them. Power of a power, multiply them. A bracket powers the number too: $(2a^2)^3 = 8a^6$.

Draw: Seven boxes marked a, split into 3 and 4, with $a^3 \times a^4 = a^7$ written under them.

M4.1.2 CHAPTER 2

Zero and negative powers

You cannot multiply by 5 no times, so what can 5^0 mean?

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** $5^0 = 0$, because there are no 5s.
- ☐ **B** $5^0 = 1$, and a pattern of dividing by 5 shows why.
- ☐ **C** $5^0 = 5$, because the 5 is still there.

IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Negative numbers, factors and powers, M3.1.7. If an index such as 2^3 looks new, read this first.
- Level 3, Fractions, decimals and percentages, M3.3.7. If 1/8 as a decimal looks new, read this first.

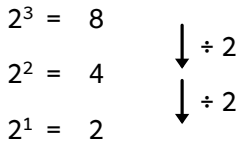
BY THE END YOU CAN

- Explain why $5^0 = 1$
- Write 2^{-3} as 1/8

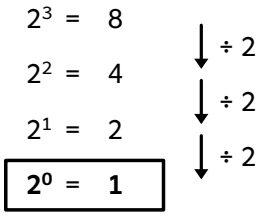
Mei writes the powers of 2 in a column: $2^3 = 8$, then $2^2 = 4$, then $2^1 = 2$. She sees a pattern. Each step down divides the answer by 2.

DRAW IT FIRST

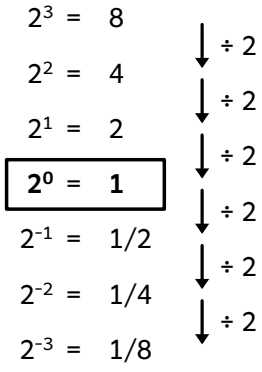
Keep going down the ladder



1 Start with $2^3 = 8$, $2^2 = 4$ and $2^1 = 2$. Each step down divides by 2.



2 Divide by 2 again. $2 \div 2 = 1$, so $2^0 = 1$.



3 Keep dividing by 2 below zero: $2^{-1} = 1/2$, $2^{-2} = 1/4$ and $2^{-3} = 1/8$.

The same thing happens for any number a except 0. $a^0 = 1$. We leave 0^0 alone, because it does not fit the pattern.

$5^2 = 25$ ↓ ÷ 5

$5^1 = 5$ ↓ ÷ 5

$\boxed{5^0 = 1}$

The same ladder for 5. One step below $5^1 = 5$ is $5 \div 5 = 1$.

The second law gives the same answer. Take $5^3 \div 5^3$. Subtracting the indices gives $3 - 3 = 0$, so it is 5^0. But any number except 0, divided by itself, is 1. So $5^0 = 1$.

WORKED EXAMPLE

Work out 7^0.

1. $7^1 \div 7^1$ has index $1 - 1 = 0$, so it equals 7^0.
2. $7 \div 7 = 1$.
3. So $7^0 = 1$.

Answer: $7^0 = 1$

Now go below zero. The ladder carries on dividing by 2, so $2^{-1} = 1/2$. A negative index means one over the positive power. $2^{-3} = 1/2^3$. The **reciprocal** of a number is 1 divided by it.

$10^3 = 1\,000$ ↓ ÷ 10

$10^2 = 100$ ↓ ÷ 10

$10^1 = 10$ ↓ ÷ 10

$\boxed{10^0 = 1}$ ↓ ÷ 10

$10^{-1} = 1/10$ ↓ ÷ 10

$10^{-2} = 1/100$ ↓ ÷ 10

$10^{-3} = 1/1\,000$

Powers of 10 going down. $10^{-3} = 1/1\,000$, which is 0.001 as a decimal.

WORKED EXAMPLE

Write 2^{-3} as a fraction.

1. A negative index means 1 divided by the positive power.
2. 2^3 means $2 \times 2 \times 2 = 8$.
3. So $2^{-3} = 1/8$.

Answer: $2^{-3} = 1/8$

WORKED EXAMPLE

Write $5^4 \div 5^6$ as a fraction.

1. Subtract the indices: $4 - 6 = -2$, so it is 5^{-2}.
2. A negative index means one over: $5^{-2} = 1/5^2$.
3. $5^2 = 25$.
4. So $5^4 \div 5^6 = 1/25$.

Answer: $5^4 \div 5^6 = 1/25$

SEE IT THREE WAYS

With things

Fold a long strip of paper in half. Fold it in half again, and once more. Open it out.

As a picture



2^{-3} of the strip

In symbols

Each fold halves the part: $1/2 \times 1/2 \times 1/2 = 1/8$. That is $2^{-3} = 1/8$.

TRY IT

Fold and count

You need: a long strip of paper, a pencil.

1. Fold the strip in half and open it. Count the parts.
2. Fold it in half twice and open it. Count again.
3. Fold it in half three times and open it. Count again.

What you will see: One fold gives 2 equal parts, two folds give 4, and three folds give 8. One part is 1/2, then 1/4, then 1/8 of the strip.

If you cannot try it, this is what happens.

Folds	Parts	One part
0	1	1
1	2	1/2
2	4	1/4
3	8	1/8

A WRONG IDEA MANY PEOPLE HAVE

~~**$2^{-3} = -8$, because of the minus sign.**~~

Why it feels right: A minus sign usually makes a number negative.

The test that breaks it: Follow the ladder: $2^1 = 2$, $2^0 = 1$, $2^{-1} = 1/2$. The numbers get smaller, but they stay above 0. They never reach −8.

What is true: A negative index means one over the positive power. $2^{-3} = 1/8$, a small positive number.

A WRONG IDEA MANY PEOPLE HAVE

~~**$5^0 = 0$, because there are no 5s.**~~

Why it feels right: Zero often means nothing.

The test that breaks it: On the ladder $5^2 = 25$ and $5^1 = 5$, each step divides by 5. The next step is $5 \div 5 = 1$, not 0.

What is true: $a^0 = 1$ for every number a except 0.

KEY WORDS

reciprocal
one divided by a number. The reciprocal of 8 is 1/8.

BACK TO YOUR GUESS

The answer is B: $5^0 = 1$, and a pattern of dividing by 5 shows why.

Each step down the powers divides by 5, and $5 \div 5 = 1$. So $5^0 = 1$. It is not 0 and it is not 5. The same pattern works for any number except 0.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. What is 9^0?

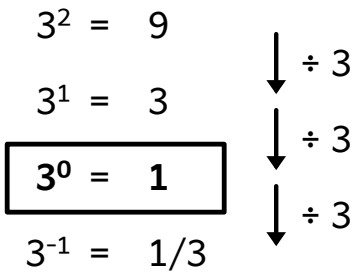
   A 1

   B 0

   C 9

   Answer at the back of the book.

2. The ladder shows powers of 3. What is 3^{-2}?



A −9

B 1/6

C 9

D 1/9

Answer at the back of the book.

3. Which number is the same as 10^{-3}?

A 0.0001

B −1 000

C 0.001

Answer at the back of the book.

4. Which sentence explains why $5^0 = 1$?

A 5^0 means no 5s are multiplied, so the answer is 0.

B The second law gives a number divided by itself the index zero, and any number except zero, divided by itself, is one.

C The index 0 does nothing, so 5^0 stays 5.

Answer at the back of the book.

5. Which is the same as $3^5 \div 3^7$?

A 1/9

B 9

C 3^{12}

Answer at the back of the book.

6. **From an earlier chapter**

   Simplify $a^9 \div a^4$. What is the index in the answer?

   Answer at the back of the book.

7. Write 2^{-3} as a decimal.

   Answer at the back of the book.

8. Work out $4^0 + 2^{-1}$. Give the answer as a decimal.

   Answer at the back of the book.

9. Which whole number n makes 1/32 equal to 2 to the power −n?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

$a^0 = 1$ for any a except 0. A negative index means one over: $2^{-3} = 1/2^3 = 1/8$. Going down the ladder divides each time. The answer stays above 0.

Draw: A ladder of powers of 2, from $2^3 = 8$ to $2^{-3} = 1/8$, with $\div 2$ arrow between each pair of rows.

Roots and fractional powers

What can it mean to raise 9 to the power 1/2?

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** 9 to the power 1/2 is half of 9, which is 4.5.
- ☐ **B** 9 to the power 1/2 is the square root of 9, which is 3.
- ☐ **C** 9 to the power 1/2 is one over 9.

IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Negative numbers, factors and powers, M3.1.7. If square roots and cube roots look new, read this first.
- Level 3, Negative numbers, factors and powers, M3.1.2. If −2 × −2 looks new, read this first.

BY THE END YOU CAN

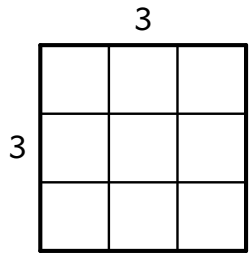
- Find the cube root of 64 and of −8
- Read 9 to the power 1/2 as √9
- Work out 8 to the power 2/3

Kofi knows that √9 = 3, because 3 × 3 = 9. Then he meets 9 to the power 1/2. What can that be?

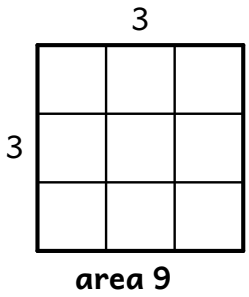
First, roots. The **square root** of a number is the positive number that gives it when multiplied by itself. The sign √ always means this positive root. Both 3 × 3 and (−3) × (−3) make 9, but √9 = 3.

DRAW IT FIRST

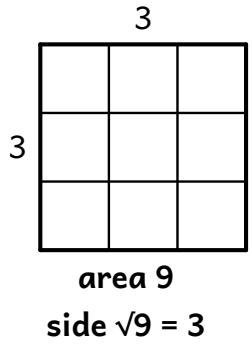
A root is *a* side



1 Lay 9 tiles in a square. It is 3 tiles along each side.



2 The area is the number of tiles: 3 × 3 = 9.

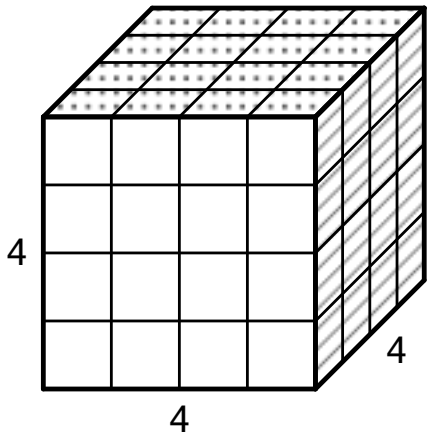


3 The square root goes back from the area to the side: √9 = 3.

Now think of a power of 1/2. We choose a meaning for 9 to the power 1/2 that keeps the laws working. Call it r, so r = 9 to the power 1/2. Squaring is a power of a power, so the third law says to multiply the indices: 1/2 × 2 = 1. So r × r = 9^1 = 9.

Both 3 and −3 give 9 when squared. We take the positive one, so 9 to the power 1/2 = √9 = 3. In general, *a* to the power 1/2 is √*a*, for *a* not below 0.

The same idea works for a cube root. A power of 1/3 cubes back to the number, so *a* to the power 1/3 is the **cube root** of *a*.



$4 \times 4 \times 4 = 64$

A cube of 64 small cubes has an edge of 4 cubes. So the cube root of 64 is 4.

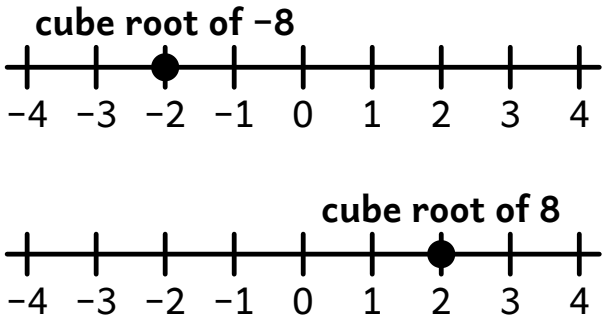
WORKED EXAMPLE

Find the cube root of 64.

1. Look for a number that gives 64 when three copies of it are multiplied together.
2. Try 3: $3 \times 3 \times 3 = 27$. That is too small.
3. Try 4: $4 \times 4 \times 4 = 64$.
4. So the cube root of 64 is 4.

Answer: The cube root of 64 is 4

A cube root can be of a negative number. Three copies of a negative number, multiplied together, give a negative number. The lines below show that the cube roots of −8 and 8 are on opposite sides of 0.



The cube root of −8 is −2, and the cube root of 8 is 2.

WORKED EXAMPLE

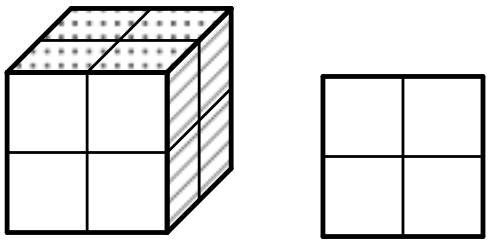
Find the cube root of −8.

1. Start with 8. 2 × 2 × 2 = 8, so the cube root of 8 is 2.
2. Try −2: (−2) × (−2) = 4, because two negatives make a positive. Then 4 × (−2) = −8.
3. So the cube root of −8 is −2.

Answer: The cube root of −8 is −2

There is a limit. In the **real numbers** you cannot take an even root of a negative number. The real numbers are all the numbers on the number line. Any number multiplied by itself is 0 or more. So there is no square root of −9. An odd root, such as a cube root, of a negative number does exist.

Now a power with a top number. 2/3 is 1/3 × 2. So 8 to the power 2/3 is the cube root of 8, then squared. The bottom number is the root. The top number is the power.



8 to the power 2/3: first the cube root, 2. Then square it: 4.

WORKED EXAMPLE

Work out 8 to the power 2/3.

1. The bottom number 3 says: take the cube root first. The cube root of 8 is 2.
2. The top number 2 says: square the result.
3. $2^2 = 2 \times 2 = 4$.
4. So 8 to the power 2/3 = 4.

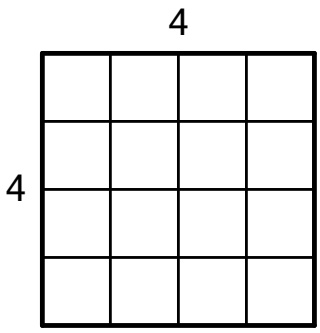
Answer: 8 to the power 2/3 = 4

SEE IT THREE WAYS

With things

A square floor has 16 tiles. How many tiles are along one side?

As a picture



In symbols

16 to the power 1/2 = √16 = 4, because 4 × 4 = 16.

TRY IT

Hunt for *a* cube root

You need: paper *and a* pencil.

1. Write the cubes of 1, 2, 3 *and* 4 in *a* list.
2. Find 27 in your list.
3. Say which number was cubed to make it.

What you will see: 3 × 3 × 3 = 27. So the cube root of 27 is 3. Reading the list backwards gives roots.

If you cannot try it, this is what happens.

Number	Its cube
1	1
2	8
3	27
4	64

A WRONG IDEA MANY PEOPLE HAVE

~~9 to the power 1/2 is half of 9, so it is 4.5.~~

Why it feels right: The index 1/2 looks like half, *and* half of 9 is 4.5.

The test that breaks it: Multiply 4.5 by itself: 4.5 × 4.5 = 20.25, not 9. But 3 × 3 = 9.

What is true: A power of 1/2 is *a* square root. 9 to the power 1/2 = √9 = 3.

A WRONG IDEA MANY PEOPLE HAVE

~~The square root of −9 is −3.~~

Why it feels right: It seems that −3 is the number that goes with 9.

The test that breaks it: Check it: (−3) × (−3) = 9, not −9. Two negatives make a positive.

What is true: No real number multiplies by itself to give −9. An even root of a negative number does not exist in the real numbers. The cube root of −8 does exist, and it is −2.

KEY WORDS

square root
the positive number that gives a number when multiplied by itself. √9 = 3.

cube root
the number that gives a number when three copies of it are multiplied together. The cube root of −8 is −2.

real numbers
all the numbers on the number line, such as −2, 0, 1/2 and 3.

BACK TO YOUR GUESS

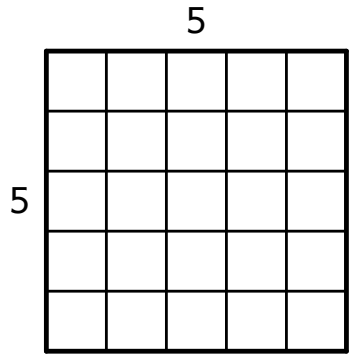
The answer is B: 9 to the power 1/2 is the square root of 9, which is 3.

A power of 1/2 is a square root. Two copies of 9 to the power 1/2 multiply to 9^1, and 3 × 3 = 9. So the answer is 3. It is not half of 9 and not one over 9.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. The picture shows a square made of 25 tiles. What is 25 to the power 1/2?



A 12.5

B 1/25

C 5

Answer at the back of the book.

2. What is the cube root of −27?

A −3

B 3

C There is no cube root.

D −9

Answer at the back of the book.

3. Why is there no square root of −16 among the real numbers?

A The answer is −4, because −4 × −4 = −16.

B The square of a real number is never negative, so no real number squares to give it.

C Negative numbers have no roots of any kind.

Answer at the back of the book.

4. Work out 64 to the power 2/3.

   A 4

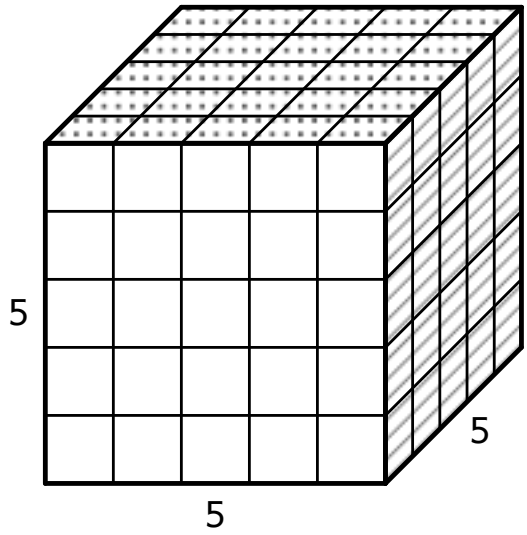
   B 128/3

   C 4 096

   D 16

   Answer at the back of the book.

5. The block is a cube of 125 small cubes. What is the cube root of 125?



   A 5

   B 25

   C 41.7

   Answer at the back of the book.

6. **From an earlier chapter**

   Write 4^{-2} as a fraction with 1 on top, such as 1/8.

   Answer at the back of the book.

7. What is the cube root of 216?

   Answer at the back of the book.

8. Work out 16 to the power 1/2 + 27 to the power 1/3.

   Answer at the back of the book.

9. A cube has a volume of 343 cm^3. How long is each edge, in cm?

Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

A power of 1/2 is a square root. A power of 1/3 is a cube root. With 2/3: root first, then power. The cube root of −8 is −2. No real square root of −9.

Draw: A square of 9 tiles with side 3, and next to it a cube of 8 small cubes with edge 2.

M4.1.4 CHAPTER 4

Standard form

Which of these is standard form for 340 000 000?

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** 3.4×10^8, because $3.4 \times 100\,000\,000$ is 340 000 000.
- ☐ **B** 34×10^7, because $34 \times 10\,000\,000$ is 340 000 000.
- ☐ **C** 0.34×10^9, because $0.34 \times 1\,000\,000\,000$ is 340 000 000.

IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Negative numbers, factors and powers, M3.1.8. If 10^6 and ten million look new, read this first.
- Level 3, Fractions, decimals and percentages, M3.3.4. If the places after the decimal point look new, read this first.

BY THE END YOU CAN

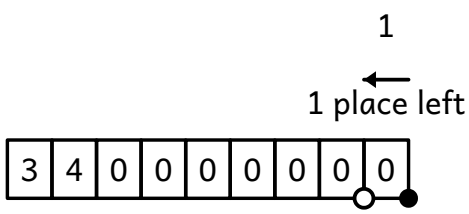
- Write 340 000 000 as 3.4×10^8
- Write 0.0005 as 5×10^{-4}
- Order numbers written in standard form

Ana reads 340 000 000 and 0.0005 in a table. Both are hard to read and hard to compare. Scientists write them in a short, fixed way called standard form.

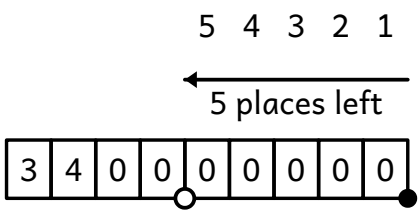
A number is in **standard form** when it is written $A \times 10^n$. Here A is at least 1 and less than 10, and n is a whole number, which can be negative.

DRAW IT FIRST

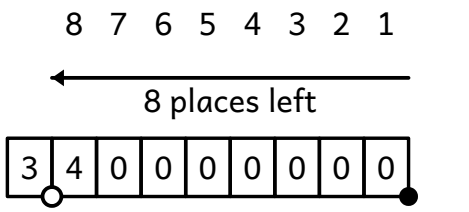
Slide the decimal point



1 The point of 340 000 000 is at the right end. Slide it one place left.

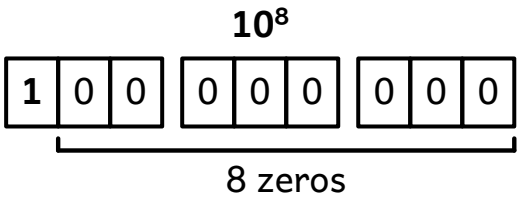


2 Slide it five places left. The digits stay where they are.



3 Slide it eight places left, until it is after the 3. Now the number reads 3.4.

The point moved 8 places left. The number got smaller by a factor of 10 each place. To keep the same size, multiply by 10 eight times. That is 10^8, a 1 followed by eight zeros.



$$340\,000\,000 = 3.4 \times 100\,000\,000 = 3.4 \times 10^8.$$

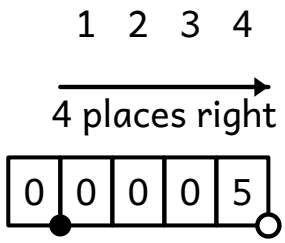
WORKED EXAMPLE

Write 340 000 000 in standard form.

1. Put the point after the first digit that is not zero: 3.4.
2. The point moved 8 places left.
3. Moving left made the number smaller, so multiply by 10^8 to put the size back.
4. So $340\,000\,000 = 3.4 \times 10^8$.

Answer: $340\,000\,000 = 3.4 \times 10^8$

Small numbers work the other way. For 0.0005 the point slides to the right, until it is after the first digit that is not zero.



0.0005 has its point moved 4 places right. That made it bigger, so the power of ten is negative.

WORKED EXAMPLE

Write 0.0005 in standard form.

1. The first digit that is not zero is 5. Put the point after it: 5.
2. The point moved 4 places right.
3. Moving right made the number bigger, so the power of ten is negative: 10^{-4}.
4. So $0.0005 = 5 \times 10^{-4}$.

Answer: $0.0005 = 5 \times 10^{-4}$

So a number of 10 or more has a positive power. A number below 1 has a negative power. A number that is at least 1 and less than 10 has the power 0, because $10^0 = 1$.

Number	Standard form
6 000	6×10^3
72 000	7.2×10^4
0.3	3×10^{-1}
0.0045	4.5×10^{-3}

Each first number is at least 1 and less than 10.

To order numbers in standard form, compare the powers of ten first. A bigger power means a bigger number. If the powers match, compare the first numbers.

WORKED EXAMPLE

Put these in order, smallest first: 6.1×10^4, 9.2×10^3, 2.5×10^4.

1. Look at the powers of ten first. The powers are 4, 3, 4.
2. The smallest power is 3, so 9.2×10^3 is the smallest number.
3. Two numbers share the power 4. Compare 6.1 and 2.5: the smaller is 2.5.
4. From smallest to largest: 9.2×10^3, 2.5×10^4, 6.1×10^4.

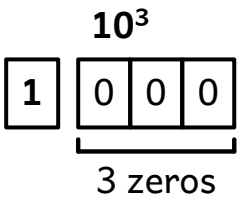
Answer: 9.2×10^3, 2.5×10^4, 6.1×10^4

SEE IT THREE WAYS

With things

Write 1 and then three zeros. That is 1 000, which is $10 \times 10 \times 10$.

As a picture



In symbols

$6\,000 = 6 \times 1\,000 = 6 \times 10^3$.

TRY IT

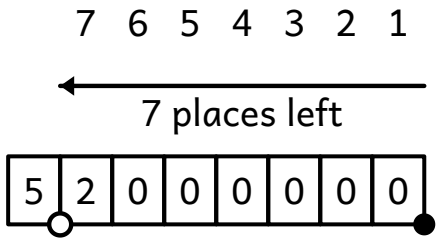
Slide the point on paper

You need: paper and a pencil.

1. Write 52 000 000 on the paper. Mark the decimal point at the right end.
2. Draw an arrow from that point to the place after the 5.
3. Count the places the point moved.

What you will see: The point moved 7 places left. So 52 000 000 = 5.2×10^7.

If you cannot try it, this is what happens.



A WRONG IDEA MANY PEOPLE HAVE

~~34×10^7 is standard form for 340 000 000, because it equals it.~~

Why it feels right: It has a power of ten, and the sum 34 × 10 000 000 does give 340 000 000.

The test that breaks it: Standard form has one rule for the first number: at least 1 and less than 10. The number 34 breaks it. So does 0.34×10^9, which is also equal to 340 000 000.

What is true: 3.4×10^8 is the one standard form. Many forms can equal a number, but only one has A at least 1 and less than 10.

A WRONG IDEA MANY PEOPLE HAVE

~~0.0005 = 5×10^4, because the point moves four places.~~

Why it feels right: The point does move four places, and 4 is the count.

The test that breaks it: Work out $5 \times 10^4 = 50\,000$. That is far bigger than 0.0005.

What is true: A number below 1 needs a negative power. $0.0005 = 5 \times 10^{-4}$.

KEY WORDS

standard form
a number written as $A \times 10^n$, with A at least 1 and less than 10, and n a whole number.

BACK TO YOUR GUESS

The answer is A: 3.4×10^8, because $3.4 \times 100\,000\,000$ is 340 000 000.

All three equal 340 000 000. Only 3.4×10^8 is standard form, because the first number must be at least 1 and less than 10. One rule means everybody writes a number the same way.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. Which is 52 000 in standard form?

   A 52×10^3

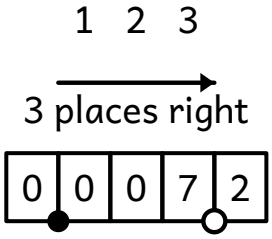
   B 5.2×10^3

   C 5.2×10^4

   D 0.52×10^5

Answer at the back of the book.

2. The picture shows the point of 0.0072 sliding. Which is 0.0072 in standard form?



A 7.2×10^{3}

B 72×10^{-4}

C 7.2×10^{-4}

D 7.2×10^{-3}

Answer at the back of the book.

3. Which of these is the largest number?

A 9.5×10^{4}

B 1.2×10^{5}

C 1.2×10^{-5}

Answer at the back of the book.

4. Which of these is NOT in standard form?

A 4.7×10^{-2}

B 1×10^{0}

C 12×10^{3}

Answer at the back of the book.

5. Which number is 3×10^{-5} written out?

A 0.00003

B 0.0003

C 300 000

Answer at the back of the book.

6. **From an earlier chapter**

   Work out 27 to the power 2/3.

   Answer at the back of the book.

7. Write 6.4×10^{-2} as a decimal.

   Answer at the back of the book.

8. How many digits does 3.4×10^{8} have when it is written out in full?

   Answer at the back of the book.

9. A virus particle is about 0.00000012 m across. Write 0.00000012 in standard form. What is the first number, A?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Standard form is $A \times 10^{n}$, with A at least 1 and below 10. Slide the point to make A and count the places. Left is a positive power, right is negative.

Draw: The digits 340000000 with the point sliding 8 places left, and 3.4×10^{8} written under it.

M4.1.5 CHAPTER 5

Calculating in standard form

How do you multiply 3×10^4 by 2×10^5 without writing out the zeros?

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** 6×10^{20}, by multiplying 3 × 2 and multiplying the powers 4 × 5.
- ☐ **B** 6×10^9, by multiplying 3 × 2 and adding the powers 4 + 5.
- ☐ **C** 5×10^9, by adding 3 + 2 and adding the powers 4 + 5.

IF THIS FEELS HARD, READ THIS FIRST

- Level 4, Powers, roots, surds and standard form, M4.1.4. If $A \times 10^n$ looks new, read the last chapter first.
- Level 3, Fractions, decimals and percentages, M3.3.5. If adding decimals such as 3.2 + 0.45 looks new, read this first.

BY THE END YOU CAN

- Multiply and divide numbers in standard form
- Add numbers in standard form by matching the powers
- Compare the size of a cell with the Earth in standard form

Hana must work out $(3 \times 10^4) \times (2 \times 10^5)$. Writing 30 000 × 200 000 is slow, and one zero too many ruins the answer.

Multiplying can be done in any order. So take the front numbers together, and take the powers of ten together.

DRAW IT FIRST

Two small jobs instead of one big one

$(3 \times 10^4) \times (2 \times 10^5)$

1 Write the two numbers: $(3 \times 10^4) \times (2 \times 10^5)$.

$(3 \times 10^4) \times (2 \times 10^5)$

$3 \times 2 = 6$

2 Multiply the front numbers: $3 \times 2 = 6$.

$(3 \times 10^4) \times (2 \times 10^5)$

$3 \times 2 = 6$ **$10^4 \times 10^5 = 10^9$**

3 Multiply the powers of ten. The first law adds the indices: $4 + 5 = 9$.

$(3 \times 10^4) \times (2 \times 10^5)$

$3 \times 2 = 6$ **$10^4 \times 10^5 = 10^9$**

$= 6 \times 10^9$

4 Put the two answers together: 6×10^9.

WORKED EXAMPLE

Work out $(3 \times 10^4) \times (2 \times 10^5)$.

1. Multiply the front numbers: $3 \times 2 = 6$.
2. Add the powers of ten: $4 + 5 = 9$, because $10^4 \times 10^5 = 10^9$.
3. So the answer is 6×10^9.

Answer: 6×10^9

Sometimes the front number comes out as 10 or more. For $(4 \times 10^3) \times (5 \times 10^6)$, the front numbers give $4 \times 5 = 20$, and the powers give 9. That is 20×10^9, which is not

standard form. Write 20 as 2×10^1. The answer is 2×10^{10}.

Dividing is the same idea. Divide the front numbers, and subtract the powers: the power on top minus the power below. Then check that the front number is at least 1 and less than 10.

WORKED EXAMPLE

Work out $(2 \times 10^7) \div (8 \times 10^2)$.

1. Divide the front numbers: $2 \div 8 = 0.25$.
2. Subtract the powers: $7 - 2 = 5$. So the quotient is 0.25×10^5.
3. 0.25 is below 1, and $0.25 = 2.5 \times 10^{-1}$.
4. So $0.25 \times 10^5 = 2.5 \times 10^{-1} \times 10^5 = 2.5 \times 10^4$.

Answer: 2.5×10^4

Adding is different. You cannot add 10^5 and 10^4 by adding the powers. First change one number so that both have the same power of ten. Then add only the front numbers.

Sum	Front	Power
×	multiply	add
÷	divide	subtract
+	add	match first
−	subtract	match first

What to do to the front numbers (Front) and to the powers of ten (Power).

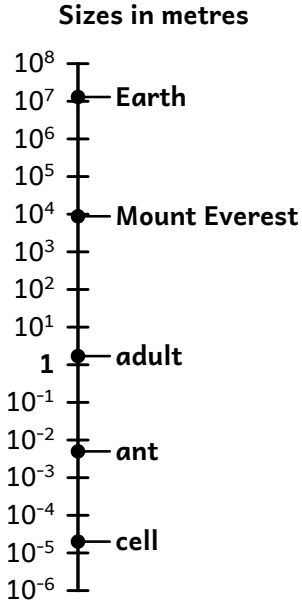
WORKED EXAMPLE

Work out $(3.2 \times 10^5) + (4.5 \times 10^4)$.

1. The powers are different. Change 4.5×10^4 so that it has the power 5.
2. 4.5×10^4 is the same as 0.45×10^5. Check: $4.5 \div 10 = 0.45$.
3. Now both have 10^5. Add the front numbers: $3.2 + 0.45 = 3.65$.
4. So the sum is 3.65×10^5, which is 365 000 in full.

Answer: 3.65×10^5

Now a big comparison. A typical animal cell is about 0.00002 m wide, which is 2×10^{-5} m. The Earth is about 1.3×10^7 m across. The scale shows both, ten times bigger at every tick.



Sizes in metres, about: cell 2×10^{-5}, ant 5×10^{-3}, adult 1.7, Mount Everest 8.8×10^3, Earth 1.3×10^7.

WORKED EXAMPLE

A cell is about 2×10^{-5} m wide and the Earth is about 1.3×10^{7} m across. About how many times wider is the Earth?

1. Divide the Earth by the cell: $(1.3 \times 10^{7}) \div (2 \times 10^{-5})$.
2. Divide the front numbers: $1.3 \div 2 = 0.65$.
3. Subtract the powers: $7 - (-5) = 12$.
4. So the quotient is 0.65×10^{12}, and $0.65 = 6.5 \times 10^{-1}$.
5. The Earth is about 6.5×10^{11} times wider than a cell.

Answer: About 6.5×10^{11} times wider

SEE IT THREE WAYS

With things

Add 3 hundreds and 4 tens. You must change the tens into hundreds first: 4 tens is 0.4 hundreds.

As a picture

Written as	Value
3×10^{2}	300
0.4×10^{2}	40
3.4×10^{2}	340

In symbols

$$3 \times 10^{2} + 4 \times 10^{1} = 3 \times 10^{2} + 0.4 \times 10^{2} = 3.4 \times 10^{2}.$$

TRY IT

Check with the full numbers

You need: paper and a pencil.

1. Write 3×10^4 as 30 000 and 2×10^5 as 200 000.
2. Multiply the front numbers, then write all the zeros that follow.
3. Compare with the standard form answer.

What you will see: $3 \times 2 = 6$, and there are $4 + 5 = 9$ zeros. So the product is 6 000 000 000, which is 6×10^9.

If you cannot try it, this is what happens.

Number	In full
3×10^4	30 000
2×10^5	200 000
6×10^9	6 000 000 000

A WRONG IDEA MANY PEOPLE HAVE

~~$3 \times 10^5 + 4 \times 10^4 = 7 \times 10^9$, because you add the front numbers and the powers.~~

Why it feels right: Adding the powers worked when we multiplied, so it feels right here too.

The test that breaks it: 3×10^5 is 300 000 and 4×10^4 is 40 000. Their sum is 340 000. But 7×10^9 is 7 000 000 000, which is far too big.

What is true: Match the powers first, then add only the front numbers. $3 \times 10^5 + 0.4 \times 10^5 = 3.4 \times 10^5$.

A WRONG IDEA MANY PEOPLE HAVE

~~$(3 \times 10^4) \times (2 \times 10^5) = 6 \times 10^{20}$, because you multiply the powers.~~

Why it feels right: The sign between the numbers is times, so it seems everything is multiplied.

The test that breaks it: Work out the sizes: 30 000 × 200 000 = 6 000 000 000, which is 6×10^9. The power 20 would give a 1 followed by 20 zeros.

What is true: Multiply the front numbers. Add the powers, because $10^4 \times 10^5 = 10^9$.

KEY WORDS

power of ten
the 10^n part of a number in standard form, such as 10^5.

BACK TO YOUR GUESS

The answer is B: 6×10^9, by multiplying 3 × 2 and adding the powers 4 + 5.

Multiply the front numbers: 3 × 2 = 6. Add the powers: 4 + 5 = 9, because $10^4 \times 10^5 = 10^9$. So the answer is 6×10^9. The front numbers are multiplied and the powers are added.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. Work out $(2 \times 10^3) \times (4 \times 10^5)$.

   A 8×10^{15}

   B 6×10^8

   C 8×10^{-2}

   D 8×10^8

Answer at the back of the book.

2. Work out $(3 \times 10^6) \div (6 \times 10^2)$.

   A 0.5×10^4

   B 5×10^3

   C 2×10^4

   D 5×10^4

   Answer at the back of the book.

3. Work out $(5 \times 10^4) \times (6 \times 10^3)$.

   A 30×10^7

   B 3×10^7

   C 3×10^8

   D 11×10^7

   Answer at the back of the book.

4. Work out $(4 \times 10^6) + (5 \times 10^5)$.

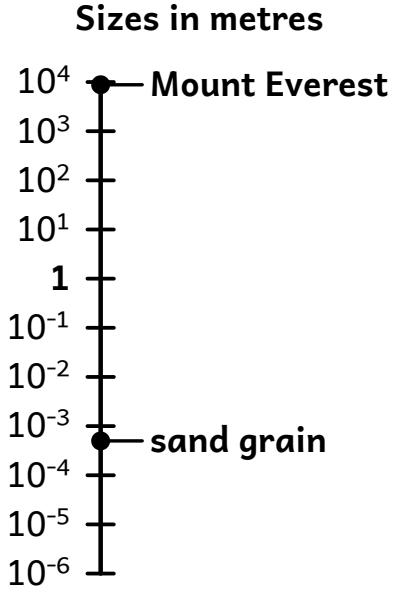
   A 4.5×10^6

   B 9×10^6

   C 9×10^{11}

   Answer at the back of the book.

5. Mount Everest is about 8.8×10^3 m high and a sand grain is about 5×10^{-4} m wide. About how many times bigger is the mountain?



A About 1.76×10^{-1} times bigger

B About 1.76×10^{-7} times bigger

C About 1.76×10^7 times bigger

Answer at the back of the book.

6. **From an earlier chapter**

Write 4 500 000 in standard form. What is the power of ten?

Answer at the back of the book.

7. Work out $(3 \times 10^8) \div (6 \times 10^3)$ in standard form. What is the power of ten?

Answer at the back of the book.

8. Work out $(2 \times 10^5) + (3 \times 10^4)$ in standard form. What is the first number?

Answer at the back of the book.

9. A row of cells is 5×10^{-3} m long. Each cell is 1×10^{-5} m wide. How many cells are in the row?

Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Multiply: multiply the front numbers, add the powers. Divide: divide, subtract. Add: match the powers first. Then check the front number is from 1 to below 10.

Draw: $(3 \times 10^4) \times (2 \times 10^5)$ in two boxes, $3 \times 2 = 6$ and $10^4 \times 10^5 = 10^9$, joined as 6×10^9.

M4.1.6 CHAPTER 6

Rational and irrational numbers

Is a decimal that never ends, such as 0.272727..., really a fraction?

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** Yes. A repeating decimal can always be written as a fraction of two whole numbers.
- ☐ **B** No. A decimal that never ends cannot be a fraction.
- ☐ **C** Only if it has fewer than ten digits.

IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Fractions, decimals and percentages, M3.3.7. If a repeating decimal such as 1/3 = 0.333... looks new, read this first.
- Level 3, Negative numbers, factors and powers, M3.1.7. If square roots such as √49 look new, read this first.

BY THE END YOU CAN

- Change a repeating decimal like 0.272727... into a fraction
- Tell rational numbers from irrational ones like √2 and π

Leila knows that 1/3 = 0.333... The 3s never stop. She asks if every decimal that never stops is a fraction.

A **rational number** can be written as a fraction of two whole numbers. The whole numbers may be negative, and the bottom number is not 0. Whole numbers are rational, because 5 = 5/1. A decimal that stops is rational, because 0.75 = 3/4.

A decimal that repeats is rational too. A **repeating decimal** has a block of digits that goes round for ever, such as 0.272727... The block 27 is the repeat.

Now the method. Call the number x, so x = 0.272727... The repeat has 2 digits, so multiply by 10^2, which is 100. Then the tails line up.

Line	Whole	Tail
100x	27	.2727...
x	0	.2727...
100x − x	27	0

The tails are the same, so taking x from 100x removes them.

DRAW IT FIRST

From a repeating decimal to a fraction

x = 0.272727...

1 Call the number x. So x = 0.272727...

x = 0.272727...

100x = 27.272727...

2 The repeat has 2 digits, so multiply by 100. That gives 100x = 27.272727...

x = 0.272727...

100x = 27.272727...

100x − x = 27

3 Take x away. The tails cancel, so 100x − x = 27. That is 99x = 27.

x = 0.272727...

100x = 27.272727...

100x − x = 27

x = 27/99 = 3/11

4 Divide by 99. So x = 27/99, which simplifies to 3/11.

WORKED EXAMPLE

Write 0.272727… as a fraction.

1. Let x = 0.272727… The repeat 27 has 2 digits, so multiply by 100.
2. 100x = 27.272727… Subtract x: 100x − x = 27.
3. 99x = 27, so x = 27/99.
4. Divide the top and bottom by 9: 27/99 = 3/11.

Answer: 3/11

Some repeats are long. 1/7 = 0.142857142857… has the 6 digit repeat 142857. Then you multiply by 10^6, which is 1 000 000, so that the tails line up.

WORKED EXAMPLE

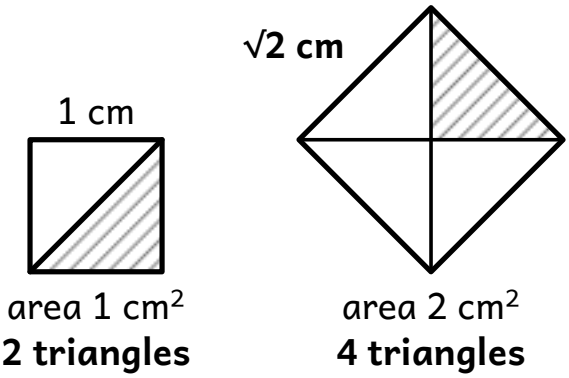
Write 0.142857142857… as a fraction.

1. Let x = 0.142857142857… The repeat 142 857 has 6 digits, so multiply by 10^6.
2. 1 000 000x − x = 142 857.142857… − 0.142857…, which is 142 857.
3. So 999 999x = 142 857, and x = 142 857/999 999.
4. Divide the top and bottom by 142 857. The bottom becomes 7, so x = 1/7.

Answer: 1/7

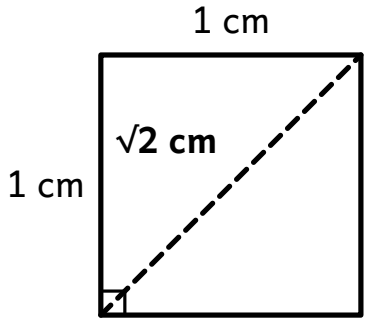
Some numbers cannot be written as a fraction at all. An **irrational number** cannot be written as a fraction of two whole numbers. The numbers √2 and π are irrational. Their decimals never end and never repeat.

Take a square with sides of 1 cm. Its area is 1 cm^2. The diagonal cuts it into 2 equal triangles. Now draw a square on the diagonal. The square on the diagonal is made of 4 of those triangles.



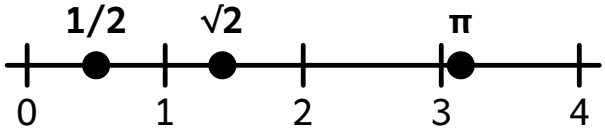
The square on the diagonal is made of 4 triangles, and the small square of 2 of them. So its area is 2 cm^2.

So the square on the diagonal has area 2 cm^2. Its side, which is the diagonal, is the positive number that gives 2 when multiplied by itself. That number is √2 cm.



The diagonal of a square with sides of 1 cm is exactly √2 cm. Mathematicians have proved that √2 is irrational.

Irrational does not mean a long decimal. 1/7 has a long decimal, but it is rational. It means no fraction of whole numbers gives the number exactly. Here are rational and irrational numbers on the number line.



1/2 is rational. √2 and π are irrational. All three sit on the line.

Here is a rule for roots. The square root of a whole number is either a whole number or irrational. Mathematicians have proved this.

WORKED EXAMPLE

Say whether each number is rational or irrational: √36, 0.2, √5, π.

1. 6 × 6 = 36, so √36 = 6. And 6 = 6/1, so √36 is rational.
2. 0.2 is rational, because 0.2 = 2/10 = 1/5.
3. 5 is not a square number, so √5 is not a whole number. So √5 is irrational. Its decimal starts like this: √5 = 2.236...
4. π = 3.14159..., which also never ends or repeats, so π is irrational.

Answer: Rational: √36 and 0.2. Irrational: √5 and π.

Number	Kind
0.75	rational
0.333...	rational
√9	rational
√2	irrational
π	irrational

0.75 = 3/4, 0.333... = 1/3 and √9 = 3 = 3/1. None of these fractions are possible for √2 or π.

SEE IT THREE WAYS

With things

Divide 1 by 3 on paper, and divide 1 by 4. See what the digits do.

As a picture

Fraction	Decimal
1/4	0.25
1/8	0.125
1/3	0.333...
2/11	0.1818...

In symbols

1/4 = 0.25 stops. 1/3 = 0.333... repeats. Every fraction of whole numbers does one or the other.

TRY IT

Divide 1 by 7 on paper

You need: paper and a pencil.

1. Divide 1 by 7 with long division. Write each digit and each remainder.
2. Carry on until the remainder is 1 again.
3. Look at the digits you wrote.

What you will see: The remainders are 3, 2, 6, 4, 5, 1. When the remainder is 1 again, the digits go round again: 0.142857142857… There are only 6 possible remainders, so a fraction must repeat or stop.

If you cannot try it, this is what happens.

Digit	Remainder
1	3
4	2
2	6
8	4
5	5
7	1

A WRONG IDEA MANY PEOPLE HAVE

~~A number with a very long decimal must be irrational.~~

Why it feels right: A decimal with many digits looks as if it will never end.

The test that breaks it: 1/7 = 0.142857142857… is long, but it is a fraction. And 0.123456789 stops, so it is 123 456 789/1 000 000 000.

What is true: Irrational means the number cannot be written as a fraction of two whole numbers. Length has nothing to do with it.

A WRONG IDEA MANY PEOPLE HAVE

~~0.999… is a little less than 1, because it never reaches 1.~~

Why it feels right: Each 9 added gets closer, but the line of 9s never seems to arrive.

The test that breaks it: Use the method. Let x = 0.999… Then 10x = 9.999… and 10x − x = 9, so 9x = 9 and x = 1.

What is true: 0.999… is exactly 1. The method shows that every repeating decimal is a fraction, and 9/9 is 1.

KEY WORDS

rational number
a number that can be written as a fraction of two whole numbers, such as 3/4 or −3/4. The bottom number is not 0.

irrational number
a number that cannot be written as a fraction of two whole numbers, such as √2 and π.

repeating decimal
a decimal with a block of digits that goes round for ever, such as 0.272727…

BACK TO YOUR GUESS

The answer is A: Yes. A repeating decimal can always be written as a fraction of two whole numbers.

A decimal that repeats can always be written as a fraction of two whole numbers. For 0.272727… it is 3/11. A decimal that never ends and never repeats cannot be, and that is what irrational means.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. Which of these numbers is irrational?

   A √49

   B √50

   C 0.333...

   Answer at the back of the book.

2. Write 0.545454... as a fraction in its simplest form.

   A 54/100

   B 54/99

   C 6/11

   Answer at the back of the book.

3. What does it mean to say a number is irrational?

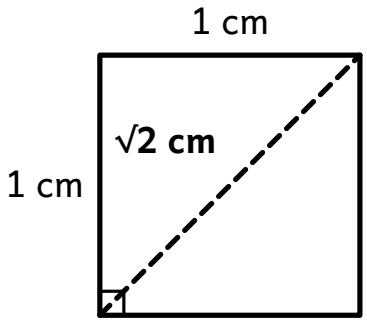
   A It cannot be written as a fraction of two whole numbers.

   B Its decimal has a lot of digits.

   C Its decimal never stops.

   Answer at the back of the book.

4. The picture shows a square with sides of 1 cm. Which statement about its diagonal is right?



   A The diagonal is 1.4 cm exactly, so it is rational.

   B The diagonal is 2 cm, because 1 + 1 = 2.

   C The diagonal is √2 cm, which cannot be written exactly as a fraction of whole numbers.

   Answer at the back of the book.

5. x = 0.272727... The repeat 27 has 2 digits. Why do we multiply by 100 and not by 10?

   A Multiplying by 10 would work as well, with a different answer.

   B The repeat has 2 digits, so shifting the point 2 places lines up the repeating tails.

   C Multiply by 2, because the repeat has 2 digits.

   Answer at the back of the book.

6. **From an earlier chapter**

   Work out $(2 \times 10^3) \times (3 \times 10^4)$ in standard form. What is the power of ten?

   Answer at the back of the book.

7. Write 0.4444... as a fraction in its simplest form.

   Answer at the back of the book.

8. How many of these numbers are irrational: $\sqrt{16}$, $\sqrt{20}$, π, 0.5, $\sqrt{7}$?

   Answer at the back of the book.

9. Let x = 0.999... What whole number is x equal to?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Rational: a fraction of whole numbers. Repeating decimals are rational: multiply by 10 to the repeat length, subtract x. Irrational: no such fraction, like $\sqrt{2}$ and π.

Draw: x = 0.2727..., then 100x = 27.2727..., then 99x = 27, and x = 3/11.

M4.1.7 CHAPTER 7

Surds

Why can √50 be written as 5√2?

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** √50 = 25√2, because 25 is a factor of 50.
- ☐ **B** √50 cannot be written any simpler.
- ☐ **C** √50 = 5√2, because 50 = 25 × 2 and √25 = 5.

IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Negative numbers, factors and powers, M3.1.6. If breaking 50 into primes looks new, read this first.
- Level 3, Negative numbers, factors and powers, M3.1.7. If square roots such as √25 look new, read this first.

BY THE END YOU CAN

- Simplify √50 to 5√2
- Add and multiply surds, like √8 + 3√2
- Rationalise the denominator of 6/√3

Yusuf finds that a square has an area of 50 cm^2. He wants its side exactly. The side is √50 cm, and a calculator only gives a rounded answer.

A **surd** is an exact root that is not a whole number or fraction, such as √2 or √50. It is left as a root. The decimal 1.414... is only an approximation of √2. And √9 is not a surd, because it is 3.

A **square factor** of a number is a factor that is a square number, such as 25 for 50. To simplify a surd, find the largest square factor, split the root, and take the root of the square part.

To simplify a surd we use one rule: for positive numbers, $\sqrt{a} \times \sqrt{b} = \sqrt{(a \times b)}$. For example $\sqrt{4} \times \sqrt{9} = 2 \times 3 = 6$, and $\sqrt{(4 \times 9)} = \sqrt{36} = 6$. Both sides agree.

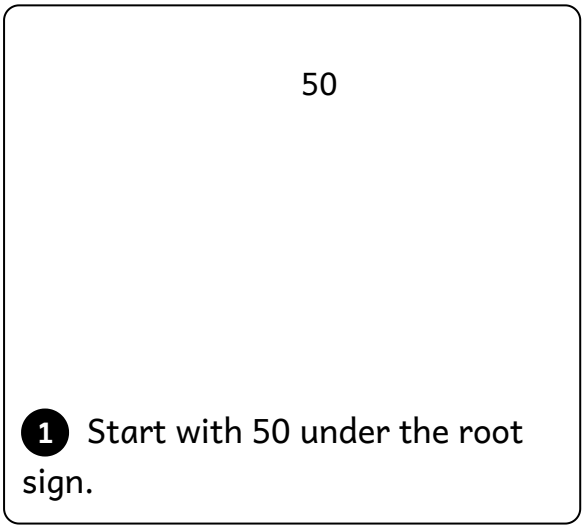
Here is why. Square the left side: $(\sqrt{a} \times \sqrt{b})^2 = \sqrt{a} \times \sqrt{a} \times \sqrt{b} \times \sqrt{b} = a \times b$. So $\sqrt{a} \times \sqrt{b}$ is the positive number whose square is $a \times b$. That number is $\sqrt{(a \times b)}$.

Sum	Value
$\sqrt{4} \times \sqrt{9}$	$2 \times 3 = 6$
$\sqrt{(4 \times 9)}$	$\sqrt{36} = 6$

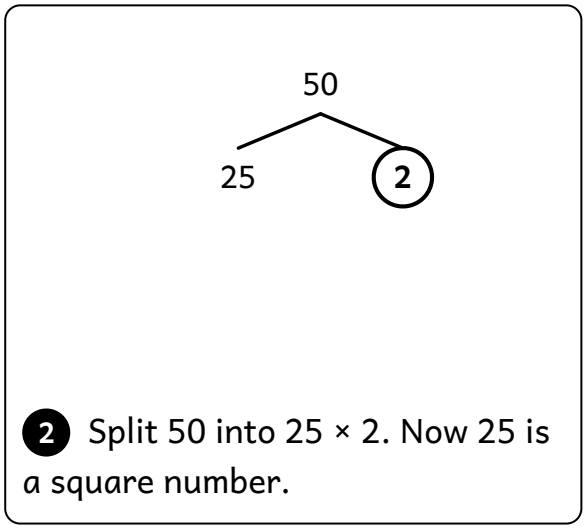
Splitting a product under a root gives the same number.

DRAW IT FIRST

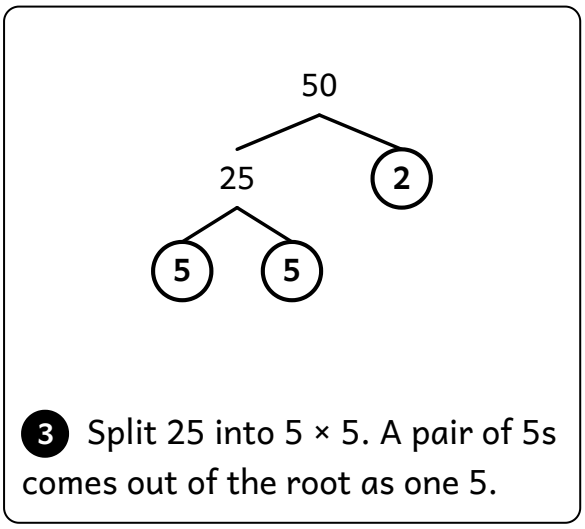
Pull a pair out of the root



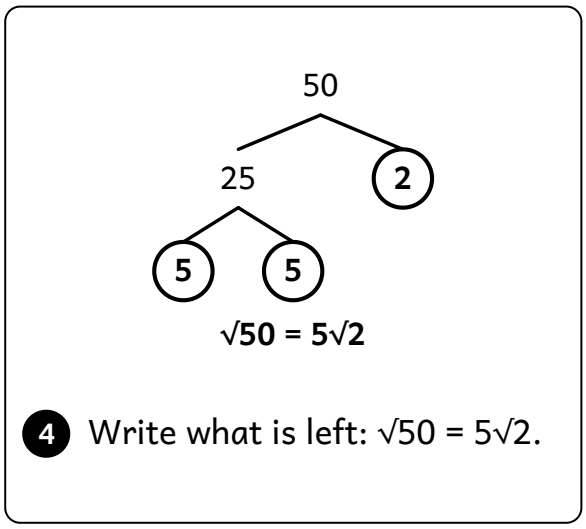
1 Start with 50 under the root sign.



2 Split 50 into 25 × 2. Now 25 is a square number.



3 Split 25 into 5 × 5. A pair of 5s comes out of the root as one 5.



4 Write what is left: $\sqrt{50} = 5\sqrt{2}$.

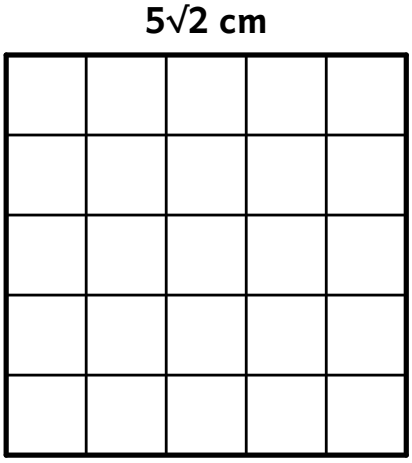
WORKED EXAMPLE

Simplify √50.

1. The largest square number that divides 50 is 25, because 50 = 25 × 2.
2. Split the root: √50 = √25 × √2.
3. √25 = 5.
4. So √50 = 5√2.

Answer: √50 = 5√2

Here is a picture of why. A square of area 50 cm^2 can be cut into 25 tiny squares, each with area 2 cm^2. Each tiny square has a side of √2 cm. There are 5 of them along each side.



Each tile has area 2 cm^2, so its side is √2 cm. Five tiles make a side of 5√2 cm.

Surd	Factor	Simplified
√8	4	2√2
√12	4	2√3
√18	9	3√2
√50	25	5√2
√75	25	5√3

Each surd with the largest square factor of its number, and its simplest form.

Now add. Surds with the same number under the root are like surds. Add them as you add like things: 2 apples and 3 apples make 5 apples. Simplify first, to see whether the surds are alike.

WORKED EXAMPLE

Simplify √8 + 3√2.

1. Simplify √8 first: √8 = √4 × √2 = 2√2.
2. Now both terms have √2: 2√2 + 3√2.
3. Add the numbers in front, as with apples: 2 + 3 = 5.
4. So √8 + 3√2 = 5√2.

Answer: √8 + 3√2 = 5√2

To multiply surds, use √a × √b = √(a × b), then simplify the result.

WORKED EXAMPLE

Simplify √6 × √3.

1. Use √a × √b = √(a × b). So √6 × √3 = √(6 × 3).
2. 6 × 3 = 18, so the product is √18.
3. Find a square factor of 18: 18 = 9 × 2.
4. √18 = √9 × √2, and √9 = 3.
5. So √6 × √3 = 3√2.

Answer: √6 × √3 = 3√2

A surd on the bottom of a fraction is awkward. To **rationalise** the denominator means to change the fraction so the bottom is a whole number. Multiply the top and the bottom by the same surd. That is multiplying by 1, so the value does not change.

WORKED EXAMPLE

Rationalise the denominator of 6/√3.

1. The bottom is √3. Multiply the top and the bottom by √3. This is the same as multiplying by 1.
2. Bottom: √3 × √3 = √9 = 3.
3. Top: 6 × √3 = 6√3. So the fraction is 6√3 ÷ 3.
4. Divide the top number by the bottom: 6 ÷ 3 = 2.
5. So 6/√3 = 2√3.
6. Check: √3 is about 1.732. Then 6 ÷ 1.732 is about 3.464, and 2 × 1.732 = 3.464.

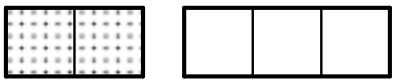
Answer: 6/√3 = 2√3

SEE IT THREE WAYS

With things

Think of √2 as one stick. Lay out 2 sticks and then 3 more sticks. Count the sticks.

As a picture



2√2 3√2

In symbols

2√2 + 3√2 = 5√2, the way 2 + 3 = 5.

TRY IT

Check with a calculator

You need: a calculator, or paper and a pencil.

1. Use √3 = 1.732, which is about right.
2. Work out 6 ÷ 1.732.
3. Work out 2 × 1.732 and compare.

What you will see: 6 ÷ 1.732 is about 3.464, and 2 × 1.732 = 3.464. They agree, so 6/√3 and 2√3 are the same number.

If you cannot try it, this is what happens.

Sum	About
6 ÷ √3	3.464
2√3	3.464

A WRONG IDEA MANY PEOPLE HAVE

~~√9 + √16 = √25, because you can add the numbers under the roots.~~

Why it feels right: The roots look like numbers that can be added inside, the way brackets work.

The test that breaks it: Work both sides out. √9 + √16 = 3 + 4 = 7, but √25 = 5. They are not equal.

What is true: You can split a product under a root, √(a × b) = √a × √b. You cannot split a sum: √(a + b) is not √a + √b.

A WRONG IDEA MANY PEOPLE HAVE

~~√50 = 25√2, because 25 is the square factor.~~

Why it feels right: The 25 is the number that divides 50, so it seems to come out in front.

The test that breaks it: Check with decimals. √50 is about 7.07. But 25√2 is about 35.4.

What is true: Only the square root of the square factor comes out. √25 = 5, so √50 = 5√2, which is about 7.07.

KEY WORDS

surd
an exact root that is not a whole number or fraction, such as √2.

square factor
a factor of a number that is itself a square number, such as 25 for 50.

rationalise
change a fraction so that its bottom is a whole number, not a surd.

BACK TO YOUR GUESS

The answer is C: √50 = 5√2, because 50 = 25 × 2 and √25 = 5.

50 = 25 × 2, and 25 is a square number with √25 = 5. The 5 comes out of the root sign and the 2 stays under it. So √50 = 5√2. It is not 25√2, because only the square root of 25 comes out.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. Simplify √72.

   A 36√2

   B 6√2

   C It cannot be simplified.

   Answer at the back of the book.

2. Simplify √12 + √27.

   A √39

   B 5√6

   C 5√3

   Answer at the back of the book.

3. Work out √2 × √8.

   A 4

   B √10

   C 16

   Answer at the back of the book.

4. Rationalise the denominator of 10/√5.

   A 10√5

   B 2√5

   C 2

   Answer at the back of the book.

5. Tariq says √36 + √64 = √100. Is he right?

   A No. √36 + √64 = 6 + 8 = 14, but √100 = 10.

   B Yes. You can add the numbers under the roots.

   C Yes, but only for square numbers.

   Answer at the back of the book.

6. **From an earlier chapter**

   Write 0.131313... as a fraction in its simplest form.

   Answer at the back of the book.

7. Work out √7 × √7.

   Answer at the back of the book.

8. Simplify √18 + √8. What is the number in front of the root?

   Answer at the back of the book.

9. Rationalise the denominator of 12/√6. What is the number in front of the root?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Find the biggest square factor: √50 = √25 × √2 = 5√2. Like surds add: 2√2 + 3√2 = 5√2. For a fraction, multiply top and bottom by the surd: 6/√3 = 2√3.

Draw: A factor tree for 50 with a ringed pair of 5s, and √50 = 5√2 written beside it.

M4.1.8 CHAPTER 8

Accuracy and bounds

> **A rod is 7.4 cm long to 1 decimal place. How long could it really be?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** Exactly 7.4 cm.
- ☐ **B** Anything from 7.3 cm to 7.5 cm.
- ☐ **C** Anything from 7.35 cm up to, but not including, 7.45 cm.

IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Fractions, decimals and percentages, M3.3.5. If rounding to the nearest whole or one decimal place looks new, read this first.
- Level 3, Perimeter, area and volume, M3.6.3. If the area of a rectangle looks new, read this first.

BY THE END YOU CAN

- Round a number to 2 or 3 significant figures
- Find the bounds of 7.4 cm rounded to 1 decimal place
- Find the largest possible area from rounded lengths

Sipho measures a rod and writes 7.4 cm, correct to 1 decimal place. He asks if the rod is exactly 7.4 cm. It almost never is. First, how rounding to significant figures works.

The **significant figures** of a number are its digits, counted from the first digit that is not zero. In 0.04567 the first significant figure is 4. We write s.f. for significant figures and d.p. for decimal places.

WORKED EXAMPLE

Round 0.04567 to 2 significant figures.

1. The first digit that is not zero is 4. It is the first significant figure.
2. Count 2 significant figures: 4 and 5. The next digit is 6.
3. The 6 is 5 or more, so round up: 0.046.

Answer: 0.046

Number	3 s.f.	2 s.f.
3 478	3 480	3 500
0.04567	0.0457	0.046
62 150	62 200	62 000

The zeros after a rounded big number hold the places. They are not significant figures.

Now back to the rod. Rounding to 1 d.p. means the numbers are measured in steps of 0.1 cm. A rod of 7.4 cm could really be anywhere that rounds to 7.4.

DRAW IT FIRST

Find the bounds of 7.4 cm

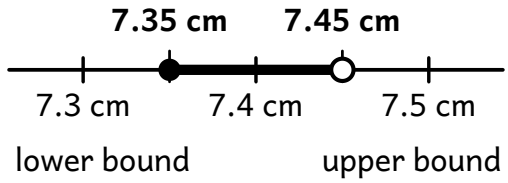
7.4 cm to 1 decimal place

1 The rod is 7.4 cm to 1 decimal place. The step between rounded numbers is 0.1 cm.

7.4 cm to 1 decimal place
half *a* step is 0.05 cm

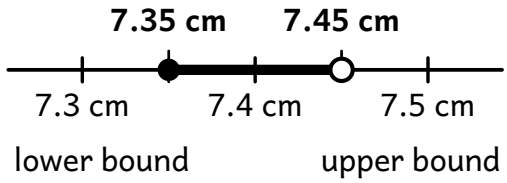
2 Half *a* step is 0.05 cm. The true length is within 0.05 cm of 7.4 cm.

7.4 cm to 1 decimal place
half *a* step is 0.05 cm



3 Mark 7.35 cm and 7.45 cm. The black dot is included and the ring is not.

7.4 cm to 1 decimal place
half *a* step is 0.05 cm



7.35 cm ≤ length < 7.45 cm

4 So the length is at least 7.35 cm and less than 7.45 cm.

Why is the upper end left out? A rod of exactly 7.45 cm would round up to 7.5 cm. And 7.35 cm rounds up to 7.4 cm, so the lower end is included. That is why we write ≤ at the bottom and < at the top.

WORKED EXAMPLE

Find the bounds of 7.4 cm rounded to 1 decimal place.

1. One decimal place means the step is 0.1 cm.
2. Half a step is 0.05 cm.
3. The lower bound is 7.4 − 0.05 = 7.35 cm.
4. The upper bound is 7.4 + 0.05 = 7.45 cm.
5. So 7.35 cm ≤ length < 7.45 cm.

Answer: 7.35 cm ≤ length < 7.45 cm

The half step depends on how the number was rounded. Here are some common ones.

SEE IT THREE WAYS

With things

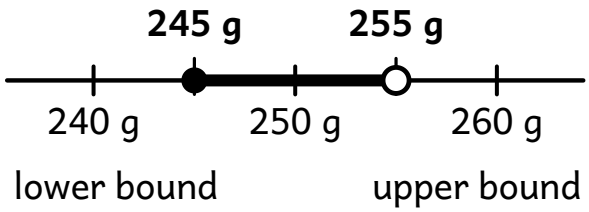
Look at a ruler marked every 0.1 cm. A true length is rounded to the nearest mark, so it can be half a mark away.

As a picture

Rounded to	Half step
1 d.p.	0.05
2 d.p.	0.005
whole number	0.5
nearest 10	5

In symbols

7.4 rounded to 1 d.p.: 7.4 − 0.05 = 7.35 and 7.4 + 0.05 = 7.45.



250 g to the nearest 10 g: at least 245 g, and less than 255 g.

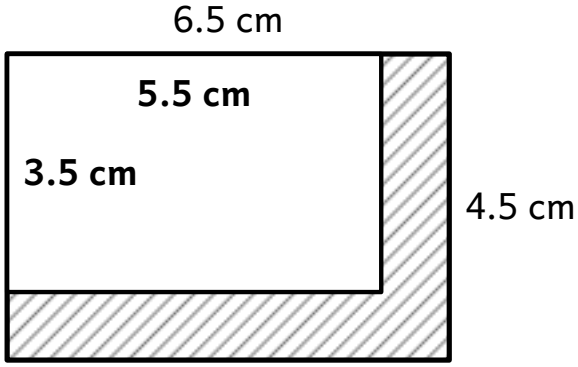
WORKED EXAMPLE

A bag weighs 250 g, to the nearest 10 g. Find the bounds of its mass.

1. The step is 10 g, so half a step is 5 g.
2. The lower bound is 250 − 5 = 245 g.
3. The upper bound is 250 + 5 = 255 g.
4. So 245 g ≤ mass < 255 g.

Answer: 245 g ≤ mass < 255 g

Now an area. A rectangle is 6 cm by 4 cm, each side to the nearest cm. Each side can be half a centimetre bigger or smaller.



The big outline uses both upper bounds. The white rectangle uses both lower bounds. The true rectangle is in between.

Area is length times width. A bigger length and a bigger width give a bigger area. So the largest area uses both upper bounds, and the smallest area uses both lower bounds.

WORKED EXAMPLE

A rectangle measures 6 cm by 4 cm, each to the nearest cm. Find the largest possible area.

1. Each side is to the nearest cm, so half a step is 0.5 cm.
2. The upper bound of 6 cm is 6 + 0.5 = 6.5 cm.
3. The upper bound of 4 cm is 4 + 0.5 = 4.5 cm.
4. The largest area uses both upper bounds: 6.5 × 4.5 = 29.25.
5. So the largest possible area is 29.25 cm^2. The true area is always just under it, so we call it the upper bound of the area.

Answer: Largest possible area: 29.25 cm^2

Each side stays just below its upper bound. So the true area can be as close to 29.25 cm^2 as you like, but it stays under it. The lower bound of the area is 19.25 cm^2, from 5.5 × 3.5. The true area is at least that.

TRY IT

Which numbers round to 7.4?

You need: paper and a pencil.

1. Write the four numbers 7.34, 7.35, 7.44, 7.45.
2. Round each one to 1 decimal place. A number halfway between rounds up.
3. Mark which ones give 7.4.

What you will see: 7.34 rounds to 7.3 and 7.35 rounds to 7.4. 7.44 rounds to 7.4 and 7.45 rounds to 7.5. The lowest that gives 7.4 is 7.35, and 7.45 is the first to give 7.5.

If you cannot try it, this is what happens.

Number	To 1 d.p.
7.34	7.3
7.35	7.4
7.44	7.4
7.45	7.5

A WRONG IDEA MANY PEOPLE HAVE

~~The upper bound of 7.4 is 7.5.~~

Why it feels right: It seems that the next number in the steps of 0.1 should be the top of the stretch.

The test that breaks it: Take 7.46. It rounds to 7.5, not 7.4. So 7.46 cannot be a length of 7.4 cm to 1 d.p.

What is true: The stretch reaches only half a step above 7.4. The upper bound is 7.45.

A WRONG IDEA MANY PEOPLE HAVE

~~0.004051 to 2 significant figures is 0.00.~~

Why it feels right: It seems that the first two digits after the point are the first two significant figures.

The test that breaks it: 0.00 is 0, and that loses all the size of the number. The zeros at the front hold places. They are not significant.

What is true: The first significant figure is 4. The first two are 4 and 0, and the next digit is 5, so round up: 0.0041.

KEY WORDS

significant figures
the digits of a number counted from the first one that is not zero.

lower bound
the smallest value a rounded number can stand for. It is included.

upper bound
the value where a rounded number stops. It is not included.

BACK TO YOUR GUESS

The answer is C: Anything from 7.35 cm up to, but not including, 7.45 cm.

Rounding to 1 decimal place means the true length is within half a step, 0.05 cm, of 7.4 cm. So it is at least 7.35 cm and less than 7.45 cm. Written as $7.35 \leq \text{length} < 7.45$.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. Round 6 284 to 2 significant figures.

   A 6 200

   B 63

   C 6 280

   D 6 300

   Answer at the back of the book.

2. A length is 12.6 m, rounded to 1 decimal place. Which line shows its bounds?

   A 12.5 m ≤ length < 12.7 m

   B 12.55 m ≤ length < 12.65 m

   C 12.6 m ≤ length < 12.7 m

   Answer at the back of the book.

3. A mass is 460 g, to the nearest 10 g. Which line shows its bounds?

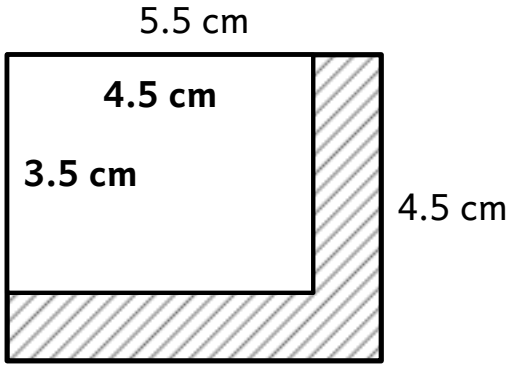
   A 455 g ≤ mass < 465 g

   B 450 g ≤ mass < 470 g

   C 460 g ≤ mass < 470 g

   Answer at the back of the book.

4. The picture shows a rectangle of 5 cm by 4 cm, each to the nearest cm. What is its largest possible area?



A 20 cm^2

B 24.75 cm^2

C 15.75 cm^2

Answer at the back of the book.

5. Which of these could NOT be a length of 7.4 cm to 1 decimal place?

A 7.35 cm

B 7.449 cm

C 7.45 cm

Answer at the back of the book.

6. **From an earlier chapter**

Write 0.00072 in standard form. What is the power of ten?

Answer at the back of the book.

7. Round 0.04587 to 3 significant figures.

Answer at the back of the book.

8. A mass is 460 g, to the nearest 10 g. What is its upper bound in g?

Answer at the back of the book.

9. A rectangle measures 6 cm by 4 cm, each to the nearest cm. What is its smallest possible area in cm^2?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Rounded values stand for a stretch: half a step each way. 7.4 to 1 d.p.: 7.35 ≤ length < 7.45. The largest area uses both upper bounds. The smallest uses both lower.

Draw: A number line with ticks 7.3, 7.4 and 7.5, a black dot at 7.35, and a ring at 7.45.

Key words, A to Z

Every key word in this book, what it means, and the chapter that teaches it.

base
the number or letter whose copies are multiplied, such as a in a^3. M4.1.1

cube root
the number that gives a number when three copies of it are multiplied together. The cube root of −8 is −2. M4.1.3

index
the small raised number. It counts the copies of the base that are multiplied. M4.1.1

indices
more than one index. M4.1.1

irrational number
a number that cannot be written as a fraction of two whole numbers, such as √2 and π. M4.1.6

lower bound
the smallest value a rounded number can stand for. It is included. M4.1.8

power of ten
the 10^n part of a number in standard form, such as 10^5. M4.1.5

rational number
a number that can be written as a fraction of two whole numbers, such as 3/4 or −3/4. The bottom number is not 0. M4.1.6

rationalise
change a fraction so that its bottom is a whole number, not a surd. M4.1.7

real numbers
all the numbers on the number line, such as −2, 0, 1/2 and 3. M4.1.3

reciprocal
one divided by a number. The reciprocal of 8 is 1/8. M4.1.2

repeating decimal
a decimal with a block of digits that goes round for ever, such as 0.272727… M4.1.6

significant figures

the digits of a number counted from the first one that is not zero. M4.1.8

square factor

a factor of a number that is itself a square number, such as 25 for 50. M4.1.7

square root

the positive number that gives a number when multiplied by itself. $\sqrt{9} = 3$. M4.1.3

standard form

a number written as $A \times 10^n$, with A at least 1 and less than 10, and n a whole number. M4.1.4

surd

an exact root that is not a whole number or fraction, such as $\sqrt{2}$. M4.1.7

upper bound

the value where a rounded number stops. It is not included. M4.1.8

Answers

M4.1.1. Laws of powers

1. **a^7**

   5 a's and 2 more a's make 5 + 2 = 7 a's.

   - **If you chose A:** You multiplied the indices. When the same base is multiplied, the a's just join up, so the indices add. Go to M4.1.1.
   - **If you chose C:** You subtracted the indices. Subtracting is for division. A product has more a's than either part. Go to M4.1.1.

2. **a^4**

   2 pairs cross out. 6 − 2 = 4 a's are left.

   - **If you chose A:** You divided the indices. Dividing powers takes away a's, so the indices are subtracted. Go to M4.1.1.
   - **If you chose B:** You added the indices. Adding is for multiplying. Here the a's below cancel some of the a's above. Go to M4.1.1.

3. **a^6**

   2 groups of 3 a's make 2 × 3 = 6 a's.

   - **If you chose B:** You added the indices. Here a^3 is written twice and the two copies are multiplied: that is 2 groups of 3 a's. Go to M4.1.1.
   - **If you chose C:** You worked out 3 squared. The 2 outside the bracket counts the copies of a^3. It does not square the 3. Go to M4.1.1.

4. **$9a^8$**

   Square the number: 3 × 3 = 9. Multiply the indices: 4 × 2 = 8.

   - **If you chose A:** You squared the a^4 but not the 3. The bracket squares everything inside it, so the 3 is squared as well. Go to M4.1.1.
   - **If you chose B:** You multiplied 3 by 2. Squaring 3 means 3 × 3, which is 9. Go to M4.1.1.
   - **If you chose C:** You added 4 and 2 for the index. The index of a^4 is multiplied by the 2 outside: 4 × 2. Go to M4.1.1.

5. **No. The letters are different, so the indices cannot be added.**

   The first law needs the same base. $a^3 \times b^4$ has two bases, so it cannot be simplified.

   - **If you chose A:** The indices only join up when the base is the same. A power of a and a power of b cannot be combined this way. Go to M4.1.1.
   - **If you chose B:** Adding or multiplying indices needs one base. With two different bases the product stays as it is. Go to M4.1.1.

6. **5**

   Subtract the indices: $8 - 3 = 5$. So $a^8 \div a^3 = a^5$.

7. **6**

   Multiply first: $a^4 \times a^5 = a^9$. Then divide: $9 - 3 = 6$.

8. **4**

   Subtract the indices: $6 - 4 = 2$. So $2^6 \div 2^4 = 2^2 = 4$. Check: $64 \div 16 = 4$.

9. **14**

   First the bracket: $3 \times 4 = 12$, so $(a^4)^3 = a^{12}$. Then add: $12 + 2 = 14$.

M4.1.2. Zero and negative powers

1. **1**

   Any number except 0 to the power 0 is 1. Here $9 \div 9 = 1$.

   - **If you chose B:** You may think no copies of 9 means nothing. But each step down the ladder divides by 9, and $9 \div 9$ is 1. Go to M4.1.2.
   - **If you chose C:** The index 0 does not leave the 9 as it is. That would be the index 1. On the ladder $9^1 = 9$, and the next step down divides by 9. Go to M4.1.2.

2. **1/9**

   Divide 1/3 by 3 once more: it is 1/9. Or use the rule: $3^{-2} = 1/3^2 = 1/9$.

   - **If you chose A:** The minus sign in the index does not make the answer negative. The ladder stays above 0 as it goes down. Go to M4.1.2.
   - **If you chose B:** You worked out $3 \times 2 = 6$. The index 2 means two 3s multiplied: $3 \times 3 = 9$. Then take one over that. Go to M4.1.2.
   - **If you chose C:** A negative index flips the number upside down. 9 is the positive power, 3^2. The answer is one over that. Go to M4.1.2.

3. **0.001**

   $10^{-3} = 1/10^3 = 1/1\,000$. One thousandth is 0.001.

   - **If you chose A:** You put one zero too many. $10^{-3} = 1/1\,000$, and the 1 sits in the third place after the point. Go to M4.1.2.
   - **If you chose B:** A negative index does not make the number negative or big. It makes it small: 1/1 000. Go to M4.1.2.

4. **The second law gives a number divided by itself the index zero, and any number except zero, divided by itself, is one.**

   $5^1 \div 5^1$ has index $1 - 1 = 0$, and $5 \div 5 = 1$.

   - **If you chose A:** No copies is not the same as the answer 0. Going down the ladder divides by 5 each time, and that reaches 1. Go to M4.1.2.
   - **If you chose C:** The index 0 is a real step on the ladder. $5^1 = 5$, and one step down is $5 \div 5 = 1$. Go to M4.1.2.

5. **1/9**

   $5 - 7 = -2$, so the power is 3^{-2}. And $3^{-2} = 1/3^2 = 1/9$.

   - **If you chose B:** You took $7 - 5$ instead of $5 - 7$. The top index is smaller, so the answer is below 1: one over 9. Go to M4.1.2.
   - **If you chose C:** You added the indices. Dividing subtracts them: $5 - 7 = -2$, so the power is 3^{-2}, which is 1/9. Go to M4.1.1.

6. **5**

   Subtract the indices: $9 - 4 = 5$. So the answer is a^5.

7. **0.125**

   $2^{-3} = 1/2^3 = 1/8$. And $1 \div 8 = 0.125$.

8. **1.5**

   $4^0 = 1$ and $2^{-1} = 1/2 = 0.5$. Together: $1 + 0.5 = 1.5$.

9. **5**

   $32 = 2 \times 2 \times 2 \times 2 \times 2 = 2^5$. So $1/32 = 2^{-5}$, and n = 5.

M4.1.3. Roots and fractional powers

1. **5**

   The side of the square is 5 tiles, because 5 × 5 = 25. So 25 to the power 1/2 = √25 = 5.

   - **If you chose A:** You took half of 25. A power of 1/2 is a square root, which is the side of the square, not half the area. Go to M4.1.3.
   - **If you chose B:** One over 25 is what a negative index gives. A positive index of 1/2 gives a root. Go to M4.1.2.

2. **−3**

   (−3) × (−3) = 9, and 9 × (−3) = −27. So the cube root of −27 is −3.

   - **If you chose B:** You dropped the minus sign. 3 × 3 × 3 = 27, not −27. A negative number cubed stays negative. Go to M4.1.3.
   - **If you chose C:** Only even roots of negative numbers fail. A cube root of a negative number exists, because three negatives multiply to a negative. Go to M4.1.3.
   - **If you chose D:** You divided by 3. A cube root is the number whose three copies, multiplied together, give the number. Go to M4.1.3.

3. **The square of a real number is never negative, so no real number squares to give it.**

   Positive times positive is positive, and negative times negative is positive. Zero times zero is zero. So a square is never below 0.

   - **If you chose A:** A negative times a negative is positive: −4 × −4 = +16. So −4 does not square to −16. Go to M3.1.2.
   - **If you chose C:** Cube roots of negative numbers exist: the cube root of −8 is −2. Only even roots, such as square roots, fail. Go to M4.1.3.

4. **16**

   The cube root of 64 is 4. Squared, that is 4 × 4 = 16.

   - **If you chose A:** You took the cube root, which is 4, and stopped. The top number 2 still says to square the result. Go to M4.1.3.
   - **If you chose B:** You multiplied 64 by 2/3. A power is not a multiplier. The bottom number is a root and the top number is a power. Go to M4.1.3.
   - **If you chose C:** You squared 64 and ignored the bottom number. The bottom number 3 says to take a cube root as well. Go to M4.1.3.

5. **5**

   $5 \times 5 \times 5 = 125$, so the edge is 5 small cubes long.

   - **If you chose B:** 25 is the number of small cubes in one layer, 5×5. The cube root is the edge, the number you multiply three times: $5 \times 5 \times 5$. Go to M4.1.3.
   - **If you chose C:** You divided 125 by 3. The cube root is the edge of the cube, the number you multiply three times. Go to M4.1.3.

6. **1/16**

   $4^{-2} = 1/4^2$, and $4^2 = 16$.

7. **6**

   $6 \times 6 \times 6 = 216$, so the cube root of 216 is 6.

8. **7**

   16 to the power 1/2 = 4, because $4 \times 4 = 16$. 27 to the power 1/3 = 3, because $3 \times 3 \times 3 = 27$. Together: $4 + 3 = 7$.

9. **7**

   The edge is the number that gives 343 when multiplied three times: $7 \times 7 \times 7 = 343$. So each edge is 7 cm.

M4.1.4. Standard form

1. **5.2×10^4**

   The point moves 4 places left, from the end of 52 000 to after the 5. So $52\,000 = 5.2 \times 10^4$.

   - **If you chose A:** This equals 52 000, but 52 is not between 1 and 10. Standard form needs the first number at least 1 and less than 10. Go to M4.1.4.
   - **If you chose B:** You counted the three zeros. Count the places the point moves instead: from the end of 52 000 to after the 5 is 4 places. Go to M4.1.4.
   - **If you chose D:** This equals 52 000, but 0.52 is below 1. The first number must be at least 1. Go to M4.1.4.

2. **7.2×10^{-3}**

   The point moves 3 places right, to after the 7. The number gets bigger, so the power is negative: 7.2×10^{-3}.

   - **If you chose A:** The point moved right, so the number got bigger by 10 each place. To undo that the power must be negative, not positive. Go to M4.1.4.
   - **If you chose B:** This equals 0.0072, but 72 is not between 1 and 10. Slide the point past the 7, so the first number is 7.2. Go to M4.1.4.
   - **If you chose C:** You counted all four places after the point. The point only moves to after the 7, which is 3 places, so the power is −3. Go to M4.1.4.

3. **1.2×10^{5}**

   The biggest power of ten is 5, and only one number has it. The other power of 5 has a minus sign, so that number is tiny. So 1.2×10^{5} is the largest.

   - **If you chose A:** You compared the first numbers, 9.5 and 1.2. Compare the powers of ten first. 10^{5} is ten times 10^{4}, so $1.2 \times 10^{5} = 120\,000$ beats $9.5 \times 10^{4} = 95\,000$. Go to M4.1.4.
   - **If you chose C:** The minus sign matters. 10^{-5} is 1/100 000, so 1.2×10^{-5} is a tiny number, 0.000012. Look at the sign of the power as well as its size. Go to M4.1.4.

4. **12×10^{3}**

   The first number must be at least 1 and less than 10. 12 is too big. As standard form this number is 1.2×10^{4}.

   - **If you chose A:** This is in standard form. 4.7 is at least 1 and less than 10, and a negative power is allowed. Go to M4.1.4.
   - **If you chose B:** This is in standard form. The first number can equal 1, and $10^{0} = 1$. Go to M4.1.4.

5. **0.00003**

   The power −5 slides the point 5 places left from after the 3: 0.00003.

   - **If you chose B:** The 3 is one place too far left. The power −5 puts the 3 in the fifth place after the point. Go to M4.1.4.
   - **If you chose C:** A negative power makes a small number, not a big one. This answer is 300 000. Go to M4.1.4.

6. **9**

   The bottom number 3 says: take the cube root, which is 3. The top number 2 says: square it. $3 \times 3 = 9$.

7. **0.064**

   The power −2 slides the point 2 places left from after the 6: 0.064.

8. **9**

   The power 8 slides the point 8 places right: the 3 stays first and the number is 340 000 000. That has 9 digits.

9. **1.2**

   The first digit that is not zero is 1. Put the point after it: 1.2. The point moves 7 places right, so $0.00000012 = 1.2 \times 10^{-7}$.

M4.1.5. Calculating in standard form

1. **8×10^{8}**

   $2 \times 4 = 8$ and $3 + 5 = 8$. So the answer is 8×10^{8}.

   - **If you chose A:** You multiplied the powers. When you multiply numbers with the same base, 10, you add the powers: 3 + 5 = 8. Go to M4.1.1.
   - **If you chose B:** You added the front numbers. The two numbers are multiplied, so multiply the front numbers: 2 × 4 = 8. Go to M4.1.5.
   - **If you chose C:** You subtracted the powers. That is for division. For a product, add them. Go to M4.1.5.

2. **5×10^{3}**

   $3 \div 6 = 0.5$ and $6 - 2 = 4$, so the quotient is 0.5×10^{4}. Then $0.5 = 5 \times 10^{-1}$, so the answer is 5×10^{3}.

   - **If you chose A:** This is equal to the answer, but 0.5 is below 1, so it is not standard form. Write 0.5 as 5×10^{-1} and lower the power by 1. Go to M4.1.5.
   - **If you chose C:** You divided 6 by 3. The question divides 3 by 6, which is 0.5. Go to M4.1.5.
   - **If you chose D:** You moved the point of 0.5 to make 5 but kept the power 4. When the front number is multiplied by 10, the power must go down by 1. Go to M4.1.5.

3. **3×10^8**

   $5 \times 6 = 30$ and $4 + 3 = 7$, giving 30×10^7. Write 30 as 3×10^1. Then the power is $7 + 1 = 8$: 3×10^8.

   - **If you chose A:** This is equal to the answer, but 30 is not below 10, so it is not standard form. Write 30 as 3×10^1. Go to M4.1.5.
   - **If you chose B:** You made 30 into 3 but did not raise the power. Dividing the front number by 10 means the power goes up by 1. Go to M4.1.5.
   - **If you chose D:** You added the front numbers. They are multiplied: $5 \times 6 = 30$. Go to M4.1.5.

4. **4.5×10^6**

   $5 \times 10^5 = 0.5 \times 10^6$. Then $4 + 0.5 = 4.5$, so the sum is 4.5×10^6. In full, $4\,000\,000 + 500\,000 = 4\,500\,000$.

   - **If you chose B:** You added the front numbers without matching the powers. 5×10^5 is only 0.5×10^6, not 5×10^6. Go to M4.1.5.
   - **If you chose C:** You added the powers. That is for multiplying. For adding, match the powers and then add only the front numbers. Go to M4.1.5.

5. **About 1.76×10^7 times bigger**

   Divide the sizes: $8.8 \div 5 = 1.76$ and $3 - (-4) = 7$. So the mountain is about 1.76×10^7 times bigger.

   - **If you chose A:** You added 3 and −4. A comparison of size is a division, so subtract the powers: $3 - (-4) = 7$. Go to M4.1.5.
   - **If you chose B:** You subtracted the powers the wrong way round: $-4 - 3 = -7$. Take the sand grain's power from the mountain's: $3 - (-4) = 7$. The mountain is bigger, so the power must be positive. Go to M4.1.5.

6. **6**

   The point moves 6 places left, to after the 4. So $4\,500\,000 = 4.5 \times 10^6$.

7. **4**

   $3 \div 6 = 0.5$ and $8 - 3 = 5$. The front number is below 1, so write 0.5 as 5×10^{-1}. The answer is 5×10^4.

8. **2.3**

   $3 \times 10^4 = 0.3 \times 10^5$. Then $2 + 0.3 = 2.3$, so the sum is 2.3×10^5.

9. **500**

   Divide the lengths: 5 ÷ 1 = 5 and −3 − (−5) = 2. So the number of cells is 5×10^2 = 500.

M4.1.6. Rational and irrational numbers

1. **√50**

   50 is not a square number, so √50 is not a whole number. The square root of a whole number is a whole number or irrational, so √50 is irrational.

   - **If you chose A:** 49 = 7 × 7, so √49 = 7. That is the fraction 7/1, so it is rational. Go to M4.1.6.
   - **If you chose C:** This is the repeating decimal for 1/3. It is a fraction of whole numbers, so it is rational. Go to M4.1.6.

2. **6/11**

   The repeat 54 has 2 digits, so 99x = 54. Then x = 54/99 and dividing the top and bottom by 9 gives 6/11.

   - **If you chose A:** This is 0.54, which stops. The repeat needs the bottom number 99, which is one less than 100. Go to M4.1.6.
   - **If you chose B:** This is equal to the answer, but it is not in simplest form. Divide the top and bottom by 9. Go to M4.1.6.

3. **It cannot be written as a fraction of two whole numbers.**

   That is the whole definition. The decimal of an irrational number never ends and never repeats, but the fraction test is what counts.

   - **If you chose B:** 1/7 has a long decimal and is rational. Irrational is about fractions, not about length. Go to M4.1.6.
   - **If you chose C:** A decimal that never stops can repeat, such as 0.333..., and that is rational. An irrational number has a decimal that never ends and never repeats. Go to M4.1.6.

4. **The diagonal is √2 cm, which cannot be written exactly as a fraction of whole numbers.**

   The square on the diagonal has area 2 cm^2, so the diagonal is √2 cm. √2 is irrational.

   - **If you chose A:** 1.4 × 1.4 = 1.96, not 2. A decimal like 1.4 is only close to √2. The exact value is not a fraction of whole numbers. Go to M4.1.6.
   - **If you chose B:** The 2 is the area of the square built on the diagonal. The length of the diagonal is the side of that square, √2 cm. Go to M4.1.6.

5. **The repeat has 2 digits, so shifting the point 2 places lines up the repeating tails.**

   A shift of 2 places moves the repeat exactly one block along, so 100x and x have the same tail.

   - **If you chose A:** 10x = 2.7272... and x = 0.2727... have tails that are one digit out of step. Subtracting does not cancel them. Go to M4.1.6.
   - **If you chose C:** Multiplying by 2 gives 2x = 0.5454..., which does not move the repeat along. The point must shift 2 places, and that means multiplying by 10^2, which is 100. Go to M4.1.6.

6. **7**

   2 × 3 = 6 and 3 + 4 = 7, so the answer is 6×10^7.

7. **4/9**

   The repeat has 1 digit, so multiply by 10. Then 10x − x = 4, so 9x = 4 and x = 4/9.

8. **3**

   √16 = 4 is rational. 0.5 = 1/2 is rational. 20 and 7 are not square numbers, so √20 and √7 are irrational, and π is irrational. That makes 3.

9. **1**

   The repeat has 1 digit, so multiply by 10: 10x = 9.999... Then 10x − x = 9, so 9x = 9 and x = 1.

M4.1.7. Surds

1. **$6\sqrt{2}$**

   $72 = 36 \times 2$, so $\sqrt{72} = \sqrt{36} \times \sqrt{2} = 6\sqrt{2}$.

   - **If you chose A:** You took 36 out in front. Only the square root of the square factor comes out: $\sqrt{36} = 6$, not 36. Go to M4.1.7.
   - **If you chose C:** 72 has a square factor, 36, because $72 = 36 \times 2$. So it can be simplified. Go to M4.1.7.

2. **$5\sqrt{3}$**

   $\sqrt{12} = 2\sqrt{3}$ and $\sqrt{27} = 3\sqrt{3}$. Both are like surds, so $2\sqrt{3} + 3\sqrt{3} = 5\sqrt{3}$.

   - **If you chose A:** You added the numbers under the roots. Check: $\sqrt{12} + \sqrt{27}$ is about $3.46 + 5.20 = 8.66$, but $\sqrt{39}$ is only about 6.24. Go to M4.1.7.
   - **If you chose B:** The number under the root stays the same when you add like surds. Only the numbers in front are added. Go to M4.1.7.

3. **4**

   $\sqrt{2} \times \sqrt{8} = \sqrt{(2 \times 8)} = \sqrt{16} = 4$.

   - **If you chose B:** You added 2 and 8. For a product, multiply the numbers under the root: $2 \times 8 = 16$. Go to M4.1.7.
   - **If you chose C:** You multiplied 2 and 8 but did not take the root. $\sqrt{16} = 4$. Go to M4.1.7.

4. **$2\sqrt{5}$**

   Multiply the top and bottom by $\sqrt{5}$. The bottom becomes $\sqrt{5} \times \sqrt{5} = 5$, and the top becomes $10\sqrt{5}$. Then $10\sqrt{5} \div 5 = 2\sqrt{5}$.

   - **If you chose A:** You multiplied the top by $\sqrt{5}$ but forgot that the bottom becomes 5. Divide the top number by 5 as well: $10 \div 5 = 2$. Go to M4.1.7.
   - **If you chose C:** You cancelled 10 with 5 as if the bottom were 5. The bottom was $\sqrt{5}$. The root stays in the answer. Go to M4.1.7.

5. **No. $\sqrt{36} + \sqrt{64} = 6 + 8 = 14$, but $\sqrt{100} = 10$.**

   A root of a sum is not the sum of the roots. Only a root of a product splits.

   - **If you chose B:** Try it with numbers: $\sqrt{36} + \sqrt{64} = 6 + 8 = 14$, but $\sqrt{100} = 10$. Adding under the root is not allowed. Go to M4.1.7.
   - **If you chose C:** 36 and 64 are square numbers, and it still fails: 14 is not 10. Go to M4.1.7.

6. **13/99**

   The repeat 13 has 2 digits, so 99x = 13 and x = 13/99. It cannot be simplified.

7. **7**

   √7 × √7 = √(7 × 7) = √49 = 7.

8. **5**

   √18 = 3√2 and √8 = 2√2. Like surds add: 3 + 2 = 5, so the answer is 5√2.

9. **2**

   Multiply the top and bottom by √6. The bottom becomes 6 and the top becomes 12√6. Then 12 ÷ 6 = 2, so the answer is 2√6.

M4.1.8. Accuracy and bounds

1. **6 300**

   The first 2 significant figures are 6 and 2. The next digit is 8, so round up to 6 300.

   - **If you chose A:** You rounded down. The digit after the second significant figure is 8, which is 5 or more, so round up. Go to M4.1.8.
   - **If you chose B:** You dropped the zeros. The zeros hold the places, so 6 300 stays a number in the thousands. Go to M4.1.8.
   - **If you chose C:** This is 3 significant figures. The question asks for 2. Go to M4.1.8.

2. **12.55 m ≤ length < 12.65 m**

   Half a step is 0.05 m. 12.6 − 0.05 = 12.55 and 12.6 + 0.05 = 12.65.

   - **If you chose A:** You went a whole step of 0.1 each way. The stretch reaches only half a step, 0.05, each way. Go to M4.1.8.
   - **If you chose C:** This starts at 12.6 and runs one whole step up. Rounding goes both ways, so the stretch starts below 12.6. Go to M4.1.8.

3. **455 g ≤ mass < 465 g**

   Half a step is 5 g. 460 − 5 = 455 and 460 + 5 = 465.

   - **If you chose B:** You went a whole step of 10 each way. The stretch reaches only half a step, 5, each way. Go to M4.1.8.
   - **If you chose C:** This only goes upwards. A mass of 457 g would also round to 460 g, so the stretch starts below 460 g. Go to M4.1.8.

4. **24.75 cm^2**

   The upper bounds are 5.5 cm and 4.5 cm. 5.5 × 4.5 = 24.75.

   - **If you chose A:** This is the area of the rounded sides, 5 × 4. The largest possible area uses the upper bounds, which are bigger. Go to M4.1.8.
   - **If you chose C:** This is the smallest possible area, from the lower bounds. The largest uses the upper bounds. Go to M4.1.8.

5. **7.45 cm**

   7.45 is halfway between 7.4 and 7.5, and a halfway number rounds up. So it rounds to 7.5, not 7.4.

   - **If you chose A:** 7.35 is exactly halfway, and a halfway number rounds up to 7.4. So it can be a length of 7.4 cm. Go to M4.1.8.
   - **If you chose B:** 7.449 is below 7.45, so it rounds down to 7.4. It can be a length of 7.4 cm. Go to M4.1.8.

6. **−4**

   The point moves 4 places right, to after the 7. So 0.00072 = 7.2 × 10^{-4}.

7. **0.0459**

   The first significant figure is 4. The first 3 are 4, 5 and 8, and the next digit is 7, which is 5 or more. So round up to 0.0459.

8. **465**

   Half a step is 5 g. 460 + 5 = 465. The mass is below this value, but as near to it as you like.

9. **19.25**

   The smallest area uses both lower bounds: 5.5 cm and 3.5 cm. 5.5 × 3.5 = 19.25.

About this book

For learners who can work out powers such as 2^5, find square roots, and use a letter for a number. In many schools this fits teenagers of about 14 to 16, and adults who want to refresh powers, roots and standard form can start here too. Each chapter takes one or two sessions. A learner can work alone: every exercise has its answer and the reason for it.

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