●●●●○ Level 4 · Mathematics M4.3

Pythagoras and trigonometry

Find missing sides and angles in triangles, starting with right angled ones.

Learn why the squares on the sides of a right angled triangle add up, and use that to find lengths. Then meet sine, cosine and tangent, find sides and angles with a calculator, learn the exact values for 30°, 45° and 60°, and solve any triangle with the sine rule, the cosine rule, area and bearings.

YOU ARE READY FOR THIS BOOK IF

You can solve 3x = 12 and work out √(9 + 16).

One step up	M5.3 Trigonometric functions
This book	**M4.3 Pythagoras and trigonometry**
One step down	M3.5 Angles, triangles and symmetry

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M4.3.6 Exact values

M4.3.7 The sine rule and the cosine rule

M4.3.8 Area of any triangle, and bearings

Key words, A to Z

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M4.3.1 CHAPTER 1

Pythagoras' theorem

> **A triangle has a right angle and two short sides of 3 cm and 4 cm. How long is the third side?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** 7 cm, because 3 + 4 = 7.
- ☐ **B** 5 cm, because 3 × 3 + 4 × 4 = 5 × 5.
- ☐ **C** 6 cm, because the third side is between 3 and 4, and then a bit more.

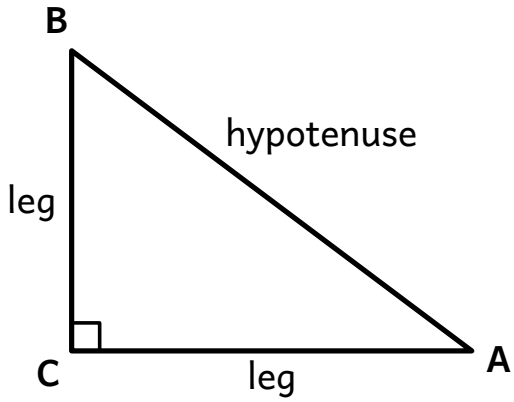
IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Negative numbers, factors and powers, M3.1.7. If squares and square roots look new, read this first.
- Level 3, Angles, triangles and symmetry, M3.5.3. If a right angled triangle looks new, read this first.

BY THE END YOU CAN

- Find the longest side of a right angled triangle
- Find a shorter side when you know the other two
- Show that sides 5, 12 and 13 make a right angle

A **right angled triangle** has one angle of 90°. The side opposite that angle is the longest side. It is called the **hypotenuse**. The two shorter sides, which meet at the right angle, are the legs.

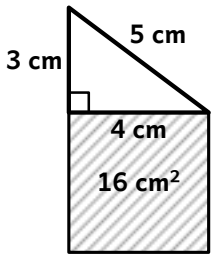


The hypotenuse faces the right angle. It is always the longest side.

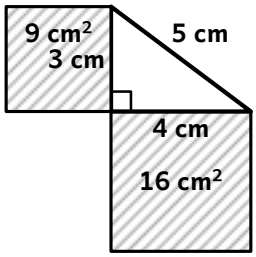
Draw a square on each side of a right angled triangle. Then compare the areas. Ana does this with a triangle whose legs are 4 cm and 3 cm.

DRAW IT FIRST

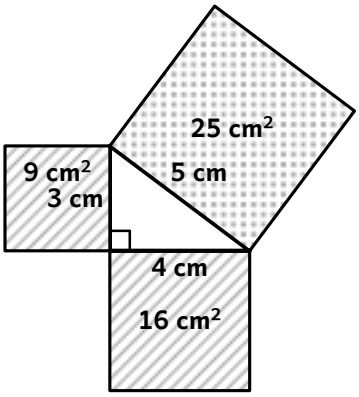
Squares on the sides



1 Draw a right angled triangle. Its legs are 4 cm and 3 cm. Draw a square on the 4 cm leg. Its area is $4 \times 4 = 16$ cm^2.



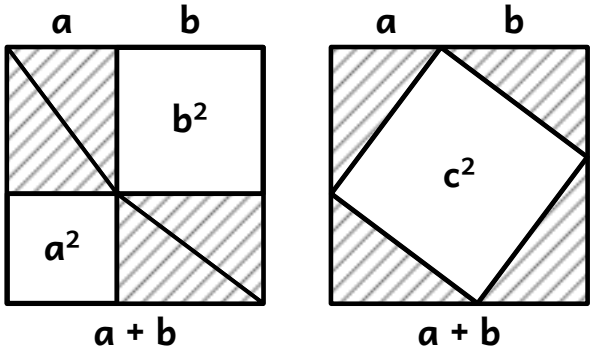
2 Draw a square on the 3 cm leg. Its area is $3 \times 3 = 9$ cm^2.



3 Draw a square on the hypotenuse. Measure the hypotenuse: it is 5 cm. The square has area $5 \times 5 = 25$ cm^2.

Look at the three areas: 16 + 9 = 25. The two small squares have the same total area as the big square. This is true for every right angled triangle. It is called **Pythagoras' theorem**.

Why is it true? Draw two squares with the same side, $a + b$. Each holds four copies of the triangle, all with legs a and b. Take the four triangles away from each square.

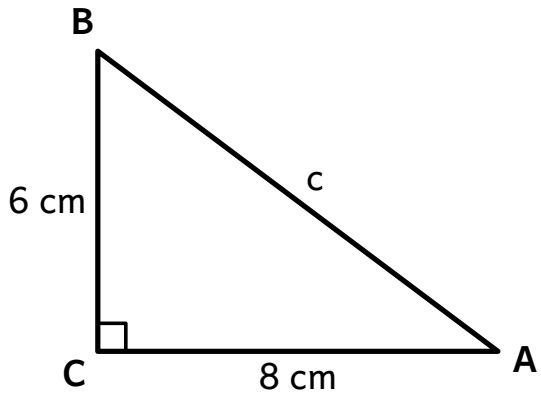


Same big square, same four triangles. So the white parts have equal area: $a^2 + b^2 = c^2$.

The big squares are equal. The four triangles are equal. So what is left must be equal too: $a^2 + b^2$ on the left, c^2 on the right. That is a proof, not a guess.

In a right angled triangle, $\mathbf{a^2 + b^2 = c^2}$. Here c is the hypotenuse. The rule works only when the triangle has a right angle.

To find the hypotenuse, add the squares of the legs, then take the square root. We take the positive root, because a length is above 0.



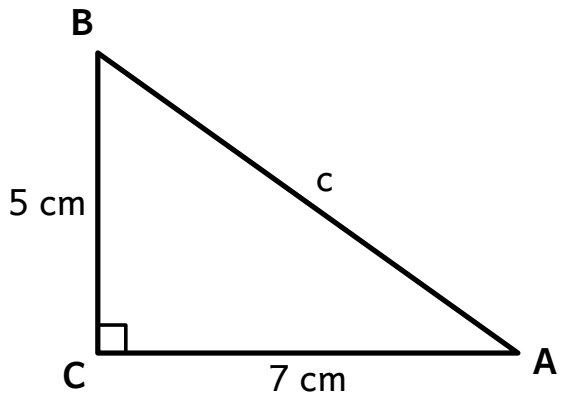
WORKED EXAMPLE

A right angled triangle has legs of 6 cm and 8 cm. Find the hypotenuse.

1. The legs are $a = 6$ and $b = 8$. We want the hypotenuse c.
2. Use $c^2 = a^2 + b^2 = 6^2 + 8^2$.
3. Square each leg and add: $36 + 64 = 100$.
4. Take the square root: $c = \sqrt{100} = 10$.

Answer: The hypotenuse is 10 cm.

The answer is not always a whole number. When it is not, the calculator gives a long decimal. Say how you rounded.



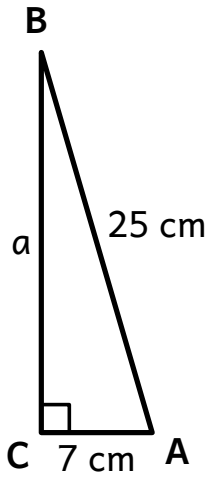
WORKED EXAMPLE

A right angled triangle has legs of 5 cm and 7 cm. Find the hypotenuse to 1 decimal place.

1. Use $c^2 = 5^2 + 7^2$.
2. Square each leg: $5^2 = 25$ and $7^2 = 49$.
3. Add: $c^2 = 25 + 49 = 74$.
4. Take the square root: $c = \sqrt{74} = 8.602$ on the calculator. To 1 decimal place that is 8.6.

Answer: To 1 decimal place, the hypotenuse is 8.6 cm.

Now the hypotenuse is known and a leg is not. Rewrite the rule. If $c^2 = a^2 + b^2$, then $a^2 = c^2 - b^2$. So you subtract. The hypotenuse is always the one you start from.



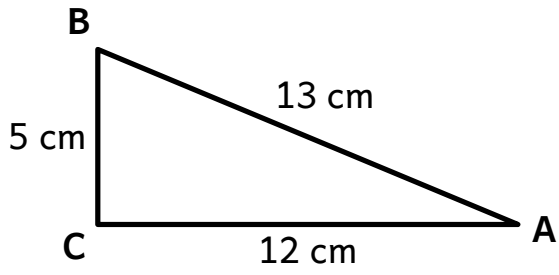
WORKED EXAMPLE

A right angled triangle has hypotenuse 25 cm and one leg 7 cm. Find the other leg.

1. The hypotenuse is c = 25. One leg is b = 7. We want the other leg a.
2. Use $a^2 = c^2 - b^2 = 25^2 - 7^2$.
3. Square each number: $25^2 = 625$ and $7^2 = 49$.
4. Subtract: $a^2 = 625 - 49 = 576$.
5. Take the square root: $a = \sqrt{576} = 24$.

Answer: The other leg is 24 cm.

The rule also works backwards. Suppose a triangle has sides 5 cm, 12 cm and 13 cm. Is there a right angle? Try the sum with c as the longest side.



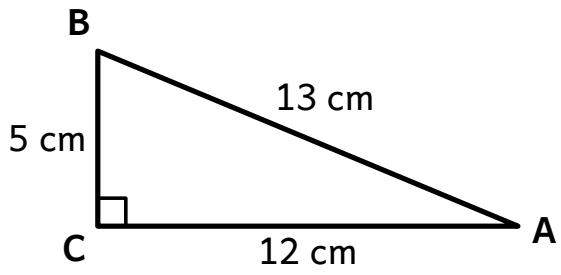
Is the corner between the 5 cm and 12 cm sides a right angle? The sums will tell us.

WORKED EXAMPLE

Do sides of 5 cm, 12 cm and 13 cm make a right angled triangle?

1. The longest side is 13. If there is a right angle, it is opposite this side, so c = 13.
2. The squares of the two shorter sides are $5^2 = 25$ and $12^2 = 144$.
3. Add them: 25 + 144 = 169.
4. The square of the longest side is $13^2 = 169$.
5. The two results are equal, so $a^2 + b^2 = c^2$ holds. The triangle has a right angle.

Answer: Yes. 25 + 144 = 169, and 13 × 13 = 169, so the triangle has a right angle.



The sums matched, so the corner opposite the 13 cm side is a right angle. The longest side faces it.

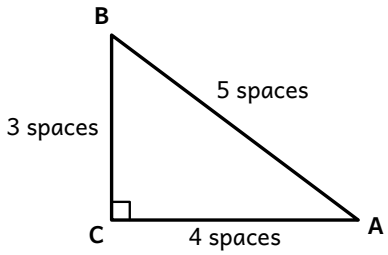
Why can we trust it? A triangle is fixed by its three sides. Build the right angled triangle with legs 5 and 12. Its hypotenuse is 13. Any triangle with sides 5, 12 and 13 is the same triangle, so it has a right angle too. The sums must match exactly. Sides 4, 5 and 6 give 16 + 25 = 41, but 6 × 6 = 36. So they make no right angle.

SEE IT THREE WAYS

With things

A builder ties a loop of string with 12 equal spaces. Pulled tight with 3 spaces, 4 spaces and 5 spaces, it makes a square corner.

As a picture



In symbols

$3^2 + 4^2 = 9 + 16 = 25 = 5^2$.

TRY IT

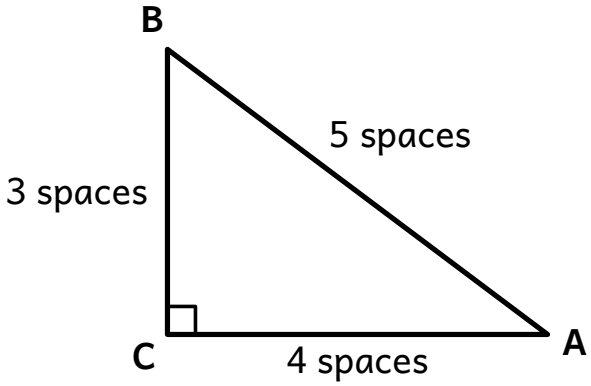
A square corner from string

You need: a piece of string about 60 cm long, a pencil and a ruler, a book with a square corner.

1. Mark the string every 5 cm, so there are 12 equal spaces. Tie the ends together.
2. Hold the loop at three marks so the sides are 3 spaces, 4 spaces and 5 spaces long. Pull it tight.
3. Put the corner of the book into the corner between the 3 and the 4. Compare.

What you will see: The corner of the string fits the corner of the book. 3, 4 and 5 make a right angle, because $3^2 + 4^2 = 5^2$.

If you cannot try it, this is what happens.



A WRONG IDEA MANY PEOPLE HAVE

~~**$a^2 + b^2 = c^2$ works for every triangle.**~~

Why it feels right: The first triangles we meet in this chapter all have right angles. So the rule seems to belong to every triangle.

The test that breaks it: Take a triangle with sides 4, 5 and 7. Then $4^2 + 5^2 = 16 + 25 = 41$, but $7^2 = 49$. The two results are not equal.

What is true: The rule holds only when the triangle has a right angle, and c is the side opposite it. When the sums do not match, there is no right angle.

A WRONG IDEA MANY PEOPLE HAVE

~~**The hypotenuse is a + b, because it joins the ends of the two legs.**~~

Why it feels right: Walking along both legs and walking across the corner look like the same trip.

The test that breaks it: In the 3, 4, 5 triangle, $a + b = 3 + 4 = 7$, but the hypotenuse is 5 cm. The short cut across is shorter than the way round.

What is true: The hypotenuse is less than $a + b$. Square first, add, then take the root: $c = \sqrt{(a^2 + b^2)}$.

KEY WORDS

hypotenuse
the longest side of a right angled triangle, opposite the right angle.

Pythagoras' theorem
in a right angled triangle, $a^2 + b^2 = c^2$, with c the hypotenuse.

BACK TO YOUR GUESS

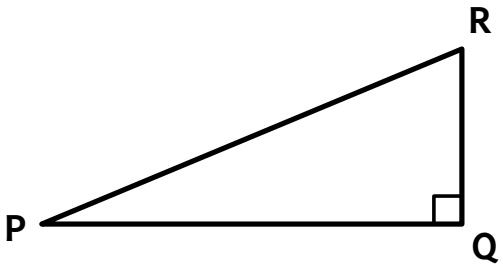
The answer is B: 5 cm, because $3 \times 3 + 4 \times 4 = 5 \times 5$.

The third side is 5 cm. It cannot be 7 cm: in a triangle, one side is always shorter than the other two added together. The sum $3 \times 3 + 4 \times 4 = 25 = 5 \times 5$ is Pythagoras' theorem, and this chapter shows why it is true.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. Which side is the hypotenuse of this triangle?



A PQ

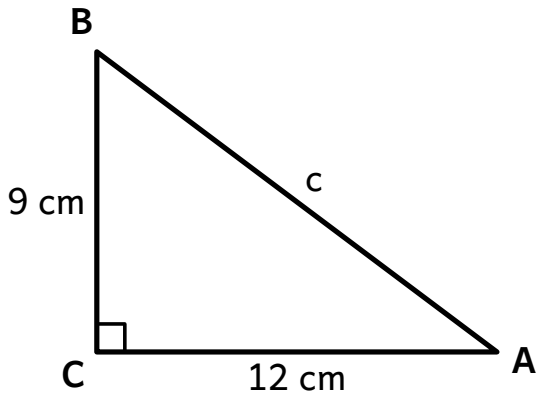
B PR

C QR

D Any side can be the hypotenuse.

Answer at the back of the book.

2. The legs of a right angled triangle are 9 cm and 12 cm. How long is the hypotenuse?



A 21 cm

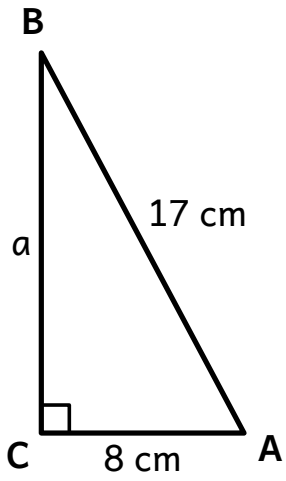
B 225 cm

C 3 cm

D 15 cm

Answer at the back of the book.

3. The hypotenuse of a right angled triangle is 17 cm and one leg is 8 cm. How long is the other leg?



A 15 cm

B 18.8 cm

C 9 cm

D 25 cm

Answer at the back of the book.

4. Which set of sides, in cm, makes a right angled triangle?

   A 5, 6, 8

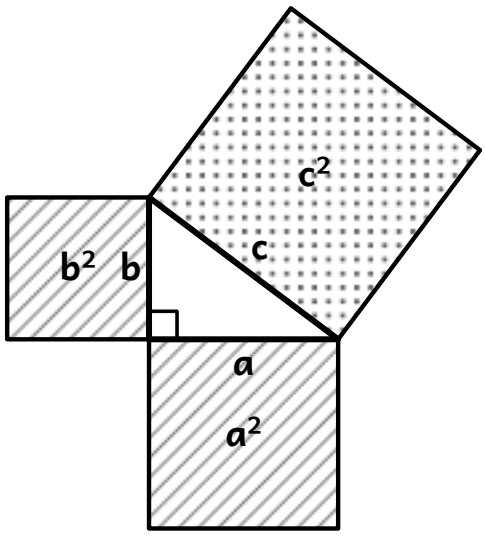
   B 4, 5, 6

   C 8, 15, 17

   D 6, 7, 10

   Answer at the back of the book.

5. The squares on the two legs of a right angled triangle have areas 36 cm^2 and 64 cm^2. What is the area of the square on the hypotenuse?



   A 10 cm^2

   B 100 cm^2

   C 14 cm^2

   Answer at the back of the book.

6. Which triangle can you use $a^2 + b^2 = c^2$ on?

   A A triangle with a right angle, where c is the side opposite it.

   B Any triangle.

   C Only a triangle whose sides are whole numbers.

   Answer at the back of the book.

7. The legs of a right angled triangle are 20 cm and 21 cm. How many centimetres is the hypotenuse?

   Answer at the back of the book.

8. The hypotenuse of a right angled triangle is 26 cm and one leg is 10 cm. How many centimetres is the other leg?

   Answer at the back of the book.

9. The legs of a right angled triangle are 4 cm and 6 cm. How long is the hypotenuse, to 1 decimal place?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

In a right angled triangle, $a^2 + b^2 = c^2$. The hypotenuse c faces the right angle. Add squares to find c. Subtract squares to find a leg.

Draw: A right angled triangle with a square on each side, labelled a^2, b^2 and c^2, and $a^2 + b^2 = c^2$ under it.

M4.3.2 CHAPTER 2

Using Pythagoras

> **A box is 2 cm by 3 cm by 6 cm. How long is the longest stick that fits inside it?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** 6 cm, because no stick can be longer than the longest edge.
- ☐ **B** 11 cm, because 2 + 3 + 6 = 11 and the stick can bend round.
- ☐ **C** 7 cm, from one bottom corner to the opposite top corner.

IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Coordinates and straight lines, M3.9.1. If the point (3, 4) on a grid looks new, read this first.
- Level 3, Perimeter, area and volume, M3.6.7. If length, width and height of a box look new, read this first.

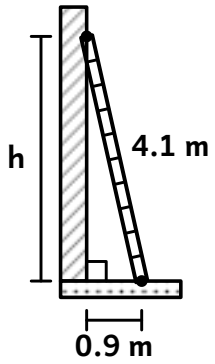
BY THE END YOU CAN

- Find how high a ladder reaches up a wall
- Find the distance between two points on a grid
- Find the longest stick that fits inside a box

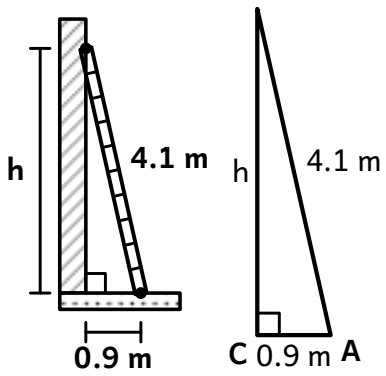
A wall stands straight up from flat ground. So the wall and the ground meet at a right angle. A ladder leaning on the wall makes a right angled triangle with them. The ladder is the hypotenuse.

DRAW IT FIRST

Find the triangle in the ladder



1 A ladder leans on a wall. Its foot is 0.9 m from the wall. Its top reaches a height h.



2 Draw the wall, the ground and the ladder as a triangle. The ladder, 4.1 m, is the hypotenuse.

You know the hypotenuse and one leg, so you subtract the squares. The height is $h^2 = 4.1^2 - 0.9^2$.

WORKED EXAMPLE

A ladder 4.1 m long stands with its foot 0.9 m from a wall. How high up the wall does it reach?

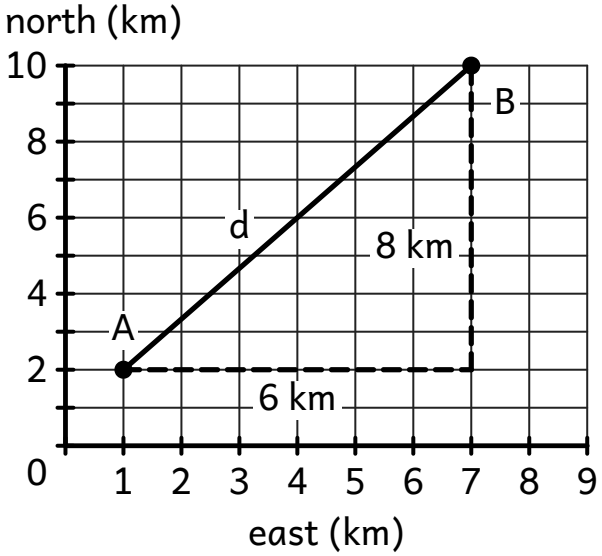
1. The ladder is the hypotenuse: c = 4.1. The foot is b = 0.9 from the wall. We want the height h.
2. Use $h^2 = c^2 - b^2 = 4.1^2 - 0.9^2$.
3. Square each number: $4.1^2 = 16.81$ and $0.9^2 = 0.81$.
4. Subtract: $h^2 = 16.81 - 0.81 = 16$.
5. Take the square root: $h = \sqrt{16} = 4$.

Answer: The ladder reaches 4 m up the wall.

Be safe. A ladder is steadiest when its foot is about 1 m from the wall for every 4 m of height. Ask someone to hold the foot, and never lean out to the

side.

A grid hides a right angled triangle too. To find how far apart two points are, go across, then go up. The two moves are the legs. The straight line between the points is the hypotenuse.



A is at (1, 2) and B is at (7, 10). One square is 1 km.

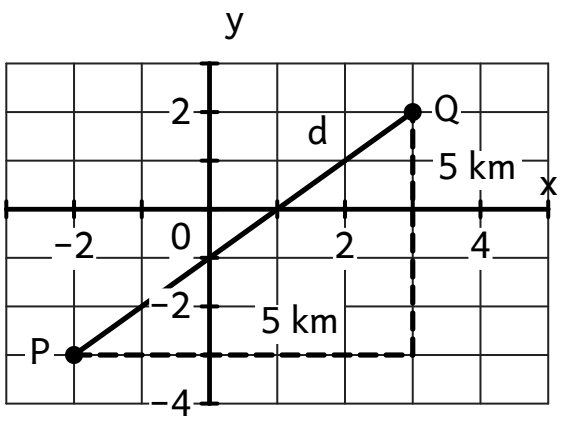
WORKED EXAMPLE

A walker goes from A (1, 2) to B (7, 10) on a map. One square is 1 km. How far apart are A and B in a straight line?

1. Across from A to B: 7 − 1 = 6 km east.
2. Up from A to B: 10 − 2 = 8 km north.
3. The two moves are the legs of a right angled triangle. The distance d is its hypotenuse.
4. $d^2 = 6^2 + 8^2 = 36 + 64 = 100$.
5. $d = \sqrt{100} = 10$ km.

Answer: The distance is 10 km.

Take care when a coordinate is negative. To go from −2 to 3 you move 3 − (−2) = 5 places. You always take the smaller number away from the larger one.



P is at (−2, −3) and Q is at (3, 2). One square is 1 km, and x and y are the kilometres east and north.

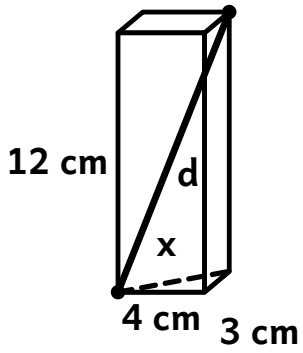
WORKED EXAMPLE

Two huts are at P (−2, −3) and Q (3, 2) on a map. One square is 1 km. How far apart are they, to 1 decimal place?

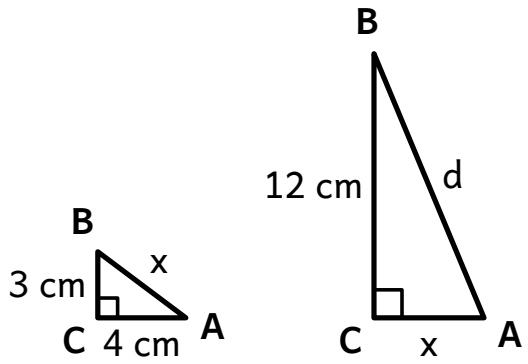
1. Across from P to Q: 3 − (−2) = 5 km.
2. Up from P to Q: 2 − (−3) = 5 km.
3. The two moves are the legs of a right angled triangle. The distance d is its hypotenuse.
4. $d^2 = 5^2 + 5^2 = 25 + 25 = 50$.
5. $d = \sqrt{50} = 7.071$ on the calculator. To 1 decimal place that is 7.1.

Answer: The huts are 7.1 km apart, to 1 decimal place.

A box has a longest stick too. It runs from a bottom corner to the opposite top corner. That line is the **space diagonal**. We find it with two right angled triangles, one flat on the floor and one standing up.



The box is drawn from the side, so the depth looks shorter than it is. The labels give the true lengths.



Triangle one finds x. Triangle two, built on x, finds d.

WORKED EXAMPLE

A box is 4 cm long, 3 cm wide and 12 cm high. How long is the longest stick that fits inside?

1. The floor is a rectangle 4 cm by 3 cm. Its diagonal x is the hypotenuse of a right angled triangle.
2. $x^2 = 4^2 + 3^2 = 16 + 9 = 25$.
3. $x = \sqrt{25} = 5$ cm.
4. Now stand a triangle on that diagonal. Its legs are x and the height, and the stick d is its hypotenuse.
5. $d^2 = 5^2 + 12^2 = 25 + 144 = 169$.
6. $d = \sqrt{169} = 13$ cm.

Answer: The longest stick is 13 cm.

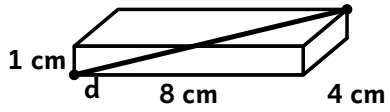
Look at the two sums together: $d^2 = x^2 + 12^2$, and $x^2 = 4^2 + 3^2$. So $d^2 = 4^2 + 3^2 + 12^2$. For any box, $d^2 = l^2 + w^2 + h^2$. You can go straight to it, but you must understand where it comes from.

SEE IT THREE WAYS

With things

Lean a pencil in a box so it touches a bottom corner and the opposite top corner. It is the longest pencil the box holds.

As a picture



In symbols

$$d^2 = l^2 + w^2 + h^2, \text{ so } d = \sqrt{(l^2 + w^2 + h^2)}.$$

TRY IT

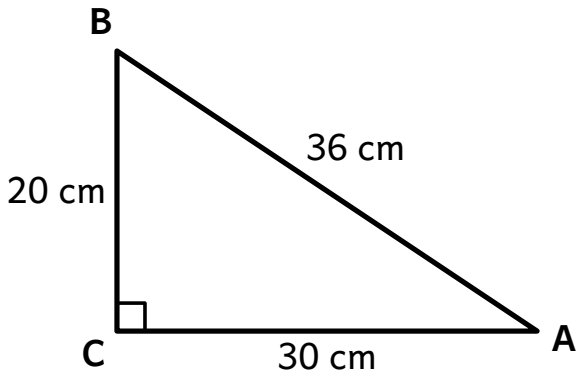
Measure a diagonal, then check it

You need: a sheet of paper, a ruler.

1. Measure the long side and the short side of the sheet. Say they are 30 cm and 20 cm.
2. Work out $30^2 + 20^2$, then take the square root.
3. Measure the diagonal, corner to corner. Compare.

What you will see: For 30 cm and 20 cm, the diagonal is √1 300, which is 36.06 cm. To the nearest centimetre it is 36 cm. Your measure should be close to the sum.

If you cannot try it, this is what happens.



A WRONG IDEA MANY PEOPLE HAVE

~~The distance between two points is the across move plus the up move.~~

Why it feels right: On a grid you count squares along the lines, and counting is adding.

The test that breaks it: For A (1, 2) and B (7, 10), across plus up is 6 + 8 = 14. Measure A to B with the squares of the grid: it is 10 squares, so 10 km.

What is true: Across plus up is the way along the two legs. The straight line is the hypotenuse, which is shorter: $d = \sqrt{(6^2 + 8^2)} = 10$.

A WRONG IDEA MANY PEOPLE HAVE

~~The longest stick in a box is as long as its longest edge.~~

Why it feels right: A stick lying along the longest edge looks as long as the box allows.

The test that breaks it: In the 4 cm by 3 cm by 12 cm box, the longest edge is 12 cm. But a 13 cm stick fits when you slant it corner to corner, because $\sqrt{169} = 13$.

What is true: The corner to corner line through the middle is longer than any edge. $d = \sqrt{(l^2 + w^2 + h^2)}$.

KEY WORDS

space diagonal
the line from one corner of a box to the opposite corner, through the inside.

BACK TO YOUR GUESS

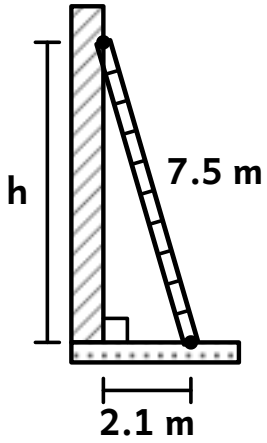
The answer is C: 7 cm, from one bottom corner to the opposite top corner.

The longest stick runs corner to corner through the middle, and it is 7 cm. The sum is $2^2 + 3^2 + 6^2 = 4 + 9 + 36 = 49$, and $\sqrt{49} = 7$. A stick cannot bend, so 11 cm is too long. This chapter uses right angled triangles to find such lengths.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. A ladder 7.5 m long stands with its foot 2.1 m from a wall. How high up the wall does it reach?



   A 7.8 m

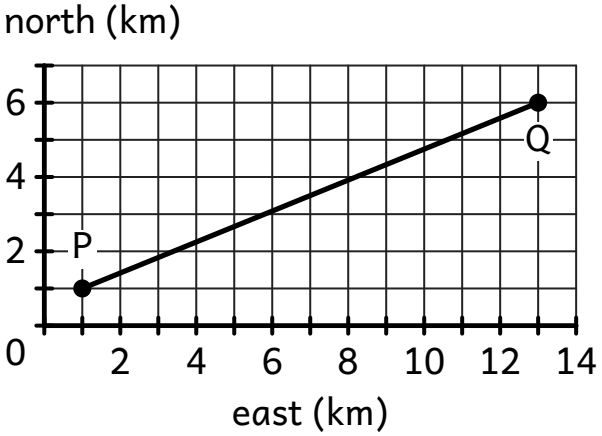
   B 5.4 m

   C 7.2 m

   D 51.84 m

   Answer at the back of the book.

2. On a map, hut P is at (1, 1) and hut Q is at (13, 6). One square is 1 km. How far apart are the huts?



A 13 km

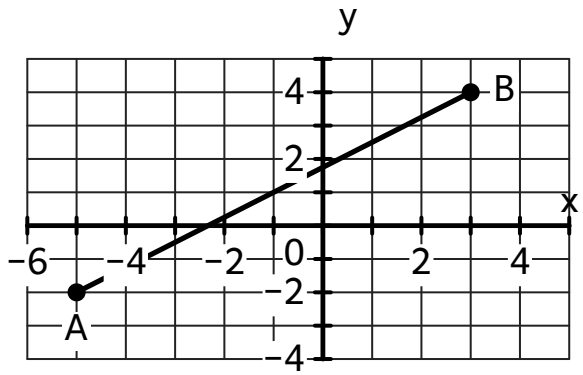
B 17 km

C 7 km

D 169 km

Answer at the back of the book.

3. On a map, A is at (−5, −2) and B is at (3, 4). One square is 1 km. How far apart are A and B?



A 2.8 km

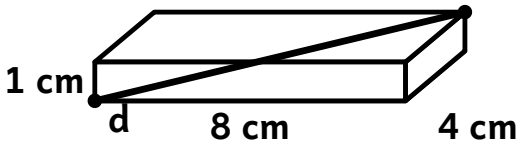
B 14 km

C 100 km

D 10 km

Answer at the back of the book.

4. A flat box is 8 cm long, 4 cm wide and 1 cm high. How long is the longest stick that fits inside?



A 8 cm

B 9 cm

C 13 cm

D 8.9 cm

Answer at the back of the book.

5. A ladder stands with its foot some distance from a wall. You know the length of the ladder and the foot distance. How do you find the height it reaches?

A Square the ladder length, take away the square of the foot distance, then take the square root.

B Square both lengths, add them, then take the square root.

C Take the foot distance away from the ladder length.

Answer at the back of the book.

6. **From an earlier chapter**

Which set of three sides, in cm, makes a right angled triangle?

A 6, 8, 9

B 9, 12, 16

C 7, 24, 25

D 5, 7, 10

Answer at the back of the book.

7. Two huts are at (2, 3) and (9, 27) on a map. One square is 1 km. How many kilometres apart are they?

Answer at the back of the book.

8. A box is 2 cm long, 4 cm wide and 4 cm high. How many centimetres long is the longest stick that fits inside?

   Answer at the back of the book.

9. A ladder 5 m long stands with its foot 1.4 m from a wall. How many metres up the wall does it reach?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Spot the right angled triangle. A ladder is the hypotenuse. Across and up on a grid are the legs. In a box, $d^2 = l^2 + w^2 + h^2$.

Draw: A box with its floor diagonal x dashed and its space diagonal d heavy, and $d^2 = l^2 + w^2 + h^2$ under it.

Opposite, adjacent and hypotenuse

> **Two right angled triangles have the same angles, but one is twice as big. Is opposite ÷ adjacent bigger in the big one?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** No, it is the same, because both sides are multiplied by the same number.
- ☐ **B** Yes, the big triangle has bigger sides, so the answer is bigger.
- ☐ **C** No, it is smaller, because dividing makes things smaller.

IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Ratio and proportion, M3.4.1. If comparing two lengths by division looks new, read this first.
- Level 3, Angles, triangles and symmetry, M3.5.3. If the angles of a triangle look new, read this first.

BY THE END YOU CAN

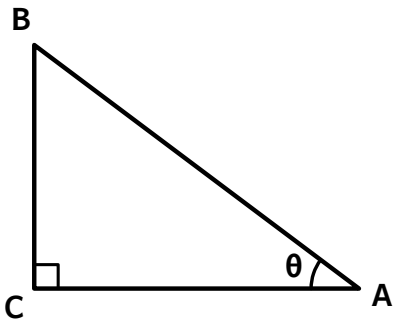
- Name the opposite, adjacent and hypotenuse for a given angle
- Show by drawing that opposite ÷ adjacent is fixed for each angle

In Chapter 1 the sides had names that did not depend on an angle. Now we pick one of the two acute angles and call it θ, the Greek letter theta. The names of two sides then depend on where θ is.

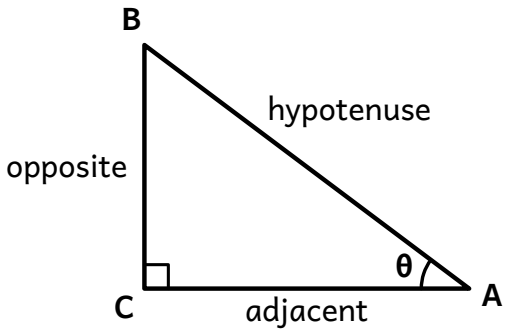
The **hypotenuse** is still the side opposite the right angle. The **opposite** side faces θ: it is across from θ and does not touch it. The ***adjacent*** side is next to θ: it touches θ and is not the hypotenuse.

DRAW IT FIRST

Name the sides for the angle θ

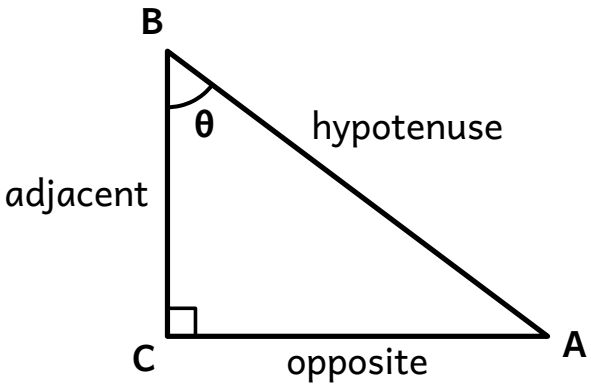


1 Pick an acute angle and mark it θ. The right angle has its small square.



2 Name the sides: hypotenuse faces the right angle, opposite faces θ, adjacent touches θ.

Choose the other angle and the names swap. The side that was opposite becomes adjacent, and the other way round. The hypotenuse never changes.



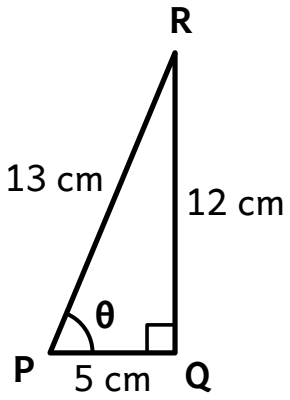
The same triangle. θ is at B now, so opposite and adjacent swap.

WORKED EXAMPLE

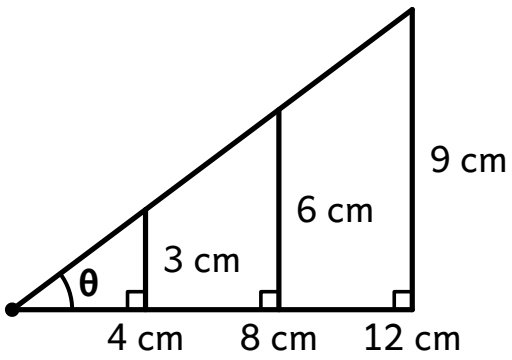
In triangle PQR the right angle is at Q. PQ is 5 cm, QR is 12 cm and PR is 13 cm. The angle θ is at P. Name the sides and find opposite ÷ adjacent.

1. The right angle is at Q. The hypotenuse faces it: PR = 13 cm.
2. θ is at P. The opposite side is across from P: QR = 12 cm.
3. The adjacent side touches P and is not the hypotenuse: PQ = 5 cm.
4. Opposite ÷ adjacent = 12 ÷ 5 = 2.4.

Answer: Opposite ÷ adjacent = 12 ÷ 5 = 2.4.



Now the big idea. Draw an angle θ. Stand a vertical line at 4 cm, at 8 cm and at 12 cm along the flat arm. You get three right angled triangles. They all have the same θ.



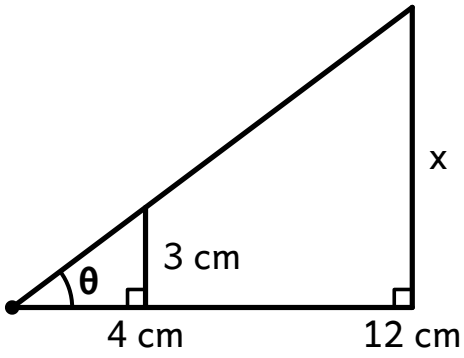
One angle θ, three triangles. Each is the one before, made bigger.

Opp cm	Adj cm	Opp ÷ adj
3	4	0.75
6	8	0.75
9	12	0.75

The last column never changes.

Why? Triangle 2 is triangle 1 enlarged by a factor of 2: every side is doubled. Triangle 3 is an enlargement by a factor of 3. When all sides are multiplied by the same number, a division of two sides stays the same. So for one angle, opposite ÷ adjacent is a fixed number. Change the angle and the number changes.

We can use the fixed number. If we know θ gives 0.75, then in any right angled triangle with that θ, the opposite side is 0.75 times the adjacent side.



WORKED EXAMPLE

A small triangle has opposite 3 cm and adjacent 4 cm for θ. A bigger triangle has the same θ and adjacent 12 cm. Find its opposite side x.

1. In the small triangle, opposite ÷ adjacent = 3 ÷ 4 = 0.75.
2. The big triangle has the same angle θ, so its opposite ÷ adjacent is also 0.75.
3. So x ÷ 12 = 0.75.
4. Multiply both sides by 12: x = 0.75 × 12 = 9.

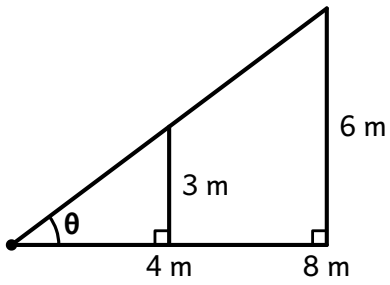
Answer: The opposite side is 9 cm.

SEE IT THREE WAYS

With things

Two ramps with the same steepness. The short one goes 4 m along and rises 3 m. The long one goes 8 m along and rises 6 m.

As a picture



In symbols

3 ÷ 4 = 6 ÷ 8 = 0.75.

TRY IT

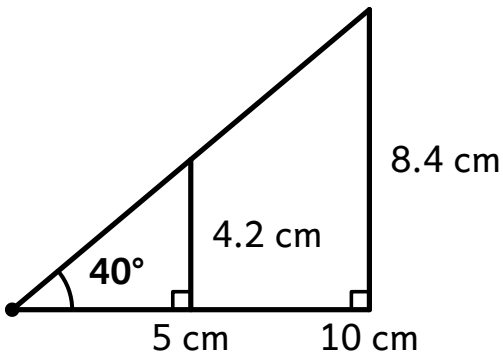
Draw two triangles with one angle

You need: paper, a ruler, a protractor, a pencil.

1. Draw a flat line 10 cm long. At its left end, use the protractor to draw a line at 40° above it.
2. Stand a vertical line at 5 cm and another at 10 cm along the flat line, up to the slanted line.
3. Measure the two vertical lines. Divide each by its flat length.

What you will see: The vertical lines measure about 4.2 cm and 8.4 cm. Both divisions give about 0.84. A bigger triangle, the same answer.

If you cannot try it, this is what happens.



A WRONG IDEA MANY PEOPLE HAVE

~~A bigger triangle with the same angle has a bigger opposite ÷ adjacent.~~

Why it feels right: Both the opposite side and the adjacent side get longer, and we think a longer side means a bigger answer.

The test that breaks it: Take 3 ÷ 4 = 0.75 and 6 ÷ 8 = 0.75. The two sides doubled, and the answer stayed at 0.75.

What is true: Doubling both sides leaves the division the same. Opposite ÷ adjacent depends only on the angle.

A WRONG IDEA MANY PEOPLE HAVE

~~The adjacent side is always the one along the bottom.~~

Why it feels right: In most pictures the flat side is called adjacent, and a position is quicker to remember than a rule.

The test that breaks it: Turn the triangle so another side is on the bottom. Opposite and adjacent have not moved: they still face and touch θ.

What is true: Opposite and adjacent depend on where θ is, not on how the page is turned. Look for the side that touches θ and is not the hypotenuse.

KEY WORDS

opposite
the side across from the angle θ, not touching it.

adjacent
the side next to the angle θ that is not the hypotenuse.

BACK TO YOUR GUESS

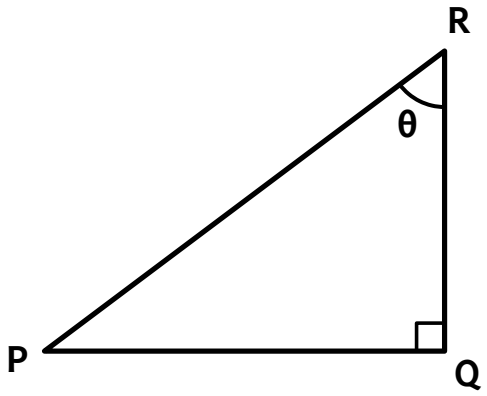
The answer is A: No, it is the same, because both sides are multiplied by the same number.

The answer is the same. In the big triangle both sides are twice as long, and 6 ÷ 8 = 3 ÷ 4. For a given angle, opposite ÷ adjacent never changes. That fixed number is the start of trigonometry.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. The angle θ is at R. Which side is opposite θ?



A PQ

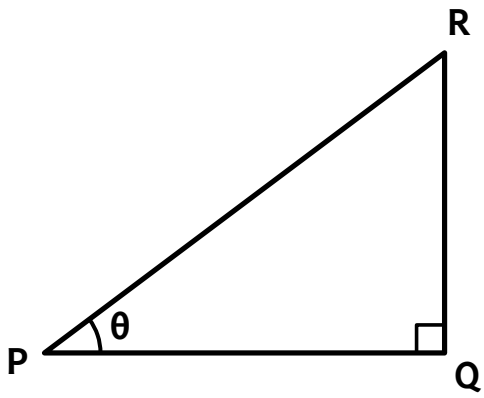
B QR

C PR

D It depends on how the page is turned.

Answer at the back of the book.

2. Now θ is at P instead. Which side is adjacent to θ?



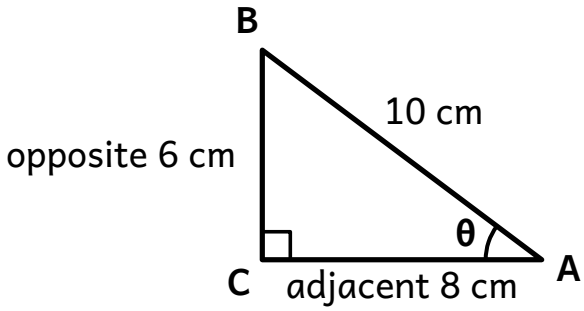
A QR

B PR

C PQ

Answer at the back of the book.

3. A right angled triangle has opposite 6 cm, adjacent 8 cm and hypotenuse 10 cm for the angle θ. What is opposite ÷ adjacent?



A 1.33

B 0.75

C 0.6

D 0.8

Answer at the back of the book.

4. Triangle A has opposite 3 cm and adjacent 5 cm for θ. Triangle B has the same θ and adjacent 15 cm. How long is the opposite side of B?

   A 13 cm

   B 5 cm

   C 45 cm

   D 9 cm

   Answer at the back of the book.

5. Two right angled triangles have the same angle θ. One is 3 times as big as the other. What is the same in both?

   A opposite ÷ adjacent

   B the length of the opposite side

   C opposite + adjacent

   Answer at the back of the book.

6. **From an earlier chapter**

   The legs of a right angled triangle are 12 cm and 16 cm. How long is the hypotenuse?

   A 28 cm

   B 4 cm

   C 20 cm

   D 400 cm

   Answer at the back of the book.

7. A small triangle has opposite 4 cm and adjacent 5 cm for θ. A bigger one has the same θ and adjacent 35 cm. How many centimetres long is its opposite side?

   Answer at the back of the book.

8. A right angled triangle has opposite 15 cm and adjacent 12 cm for θ. What is opposite ÷ adjacent, as a decimal?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Hypotenuse faces the right angle. Opposite faces θ. Adjacent touches θ. For one angle, opposite ÷ adjacent is the same in every size of triangle.

Draw: A right angled triangle with θ marked, the three sides named, and a bigger copy inside it with the same θ.

M4.3.4 CHAPTER 4

Sine, cosine and tangent

> **A ladder 5 m long leans on a wall at 75° to the ground. How high up the wall does it reach?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** About 5 m, because the ladder is 5 m long.
- ☐ **B** About 3.75 m, because 75% of 5 m is 3.75 m.
- ☐ **C** About 4.8 m, because the ladder is nearly upright.

IF THIS FEELS HARD, READ THIS FIRST

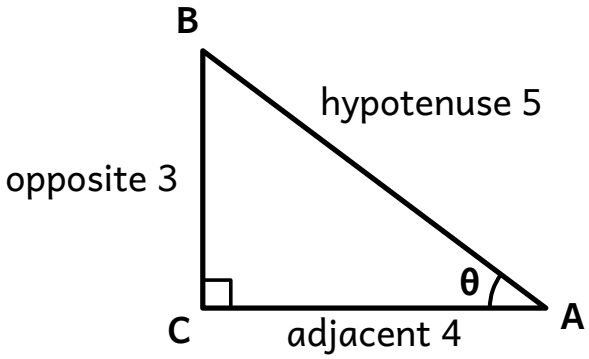
- Level 3, Ratio and proportion, M3.4.1. If dividing one length by another to compare them looks new, read this first.
- Level 3, Letters for numbers and simple equations, M3.8.5. If solving x ÷ 4 = 3 looks new, read this first.

BY THE END YOU CAN

- Choose sin, cos or tan from the sides you know
- Use sin, cos or tan to find a missing side

In the last chapter you saw that opposite ÷ adjacent is fixed for an angle θ. That fixed number has a name. It is the **tangent** of θ, written tan θ.

There are two more fixed numbers, and the same argument shows they are fixed too. The **sine** of θ is opposite ÷ hypotenuse, written sin θ. The **cosine** of θ is adjacent ÷ hypotenuse, written cos θ.



The sides are 3, 4 and 5 for this θ.

Ratio	Sides	Value
sin θ	3 ÷ 5	0.6
cos θ	4 ÷ 5	0.8
tan θ	3 ÷ 4	0.75

Three ratios of the same triangle. Each is a fixed number for this θ.

For this triangle θ is about 36.9°. Put your calculator in degrees mode, often shown as DEG. Then sin 36.87° gives about 0.6, cos 36.87° gives about 0.8 and tan 36.87° gives about 0.75. The calculator holds a table of these numbers for every angle.

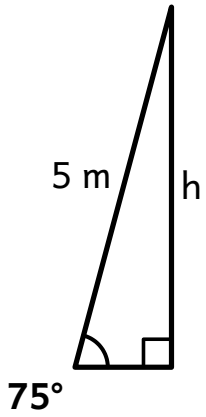
To remember the three ratios, use SOH CAH TOA. SOH: Sine is Opposite over Hypotenuse. CAH: Cosine is Adjacent over Hypotenuse. TOA: Tangent is Opposite over Adjacent.

Name	Ratio
SOH	sin = O ÷ H
CAH	cos = A ÷ H
TOA	tan = O ÷ A

To find a missing side, do three things. Label the sides O, A and H for your angle. Find the two sides that matter: the one you know and the one you want. Pick the ratio that has exactly those two letters.

Start with the ladder from the question. It makes a right angled triangle with the wall and the ground. The ladder is H, 5 m. The height h is opposite the 75° angle.

The two letters are O and H, so you use sin.



WORKED EXAMPLE

A ladder 5 m long leans on a wall at 75° to the ground. How high up the wall does it reach? Give the answer to 1 decimal place.

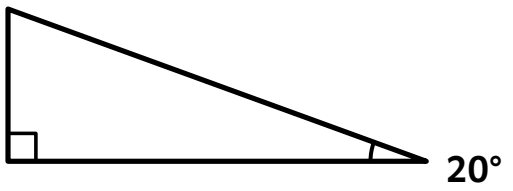
1. The ladder is the hypotenuse. The height h is opposite the 75° angle. We have O and H, so use sin.
2. sin 75° = h ÷ 5, so h = 5 × sin 75°.
3. On the calculator, to 4 decimal places, sin 75° = 0.9659.
4. Multiply: 5 × 0.9659 = 4.8295.
5. Round to 1 decimal place: h = 4.8 m.

Answer: To 1 decimal place, the ladder reaches 4.8 m.

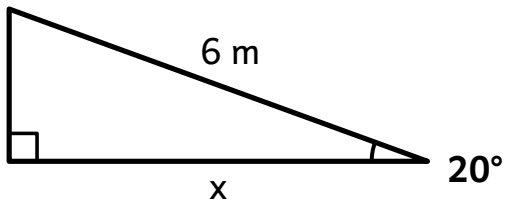
Now a case with cosine. A ramp is 6 m long and slopes at 20° to the flat ground. How far does it stretch along the ground? Draw the triangle first.

DRAW IT FIRST

Choose the ratio for a ramp



1 Draw the ramp as the slanted side. Mark the 20° angle with the ground.



2 Label the ramp 6 m, the hypotenuse. Label the flat distance x. It is adjacent to the 20° angle. We have A and H, so use cos.

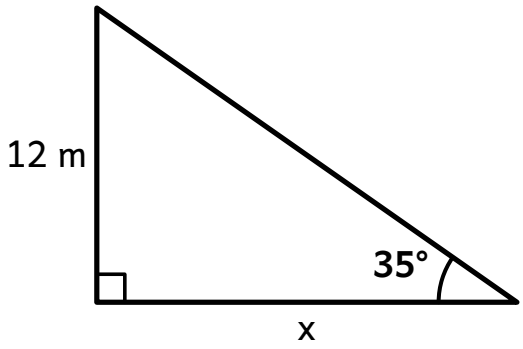
WORKED EXAMPLE

A ramp 6 m long slopes at 20° to flat ground. How far does it stretch along the ground? Give the answer to 1 decimal place.

1. The ramp is the hypotenuse. The distance x is adjacent to the 20° angle. We have A and H, so use cos.
2. cos 20° = x ÷ 6, so x = 6 × cos 20°.
3. On the calculator, to 4 decimal places, cos 20° = 0.9397.
4. Multiply: 6 × 0.9397 = 5.6382.
5. Round to 1 decimal place: x = 5.6 m.

Answer: To 1 decimal place, the ramp stretches 5.6 m.

Sometimes the side you want is on the bottom of the ratio. Then you divide. A tree is 12 m tall. From a point on level ground, the top is seen at an angle of 35° above the ground. How far is the point from the foot of the tree?



WORKED EXAMPLE

The top of a tree 12 m tall is seen at 35° above the ground. How far is the viewer from the foot of the tree? Give the answer to 1 decimal place.

1. The 12 m height is opposite the 35° angle. The distance x is adjacent. We have O and A, so use tan.
2. tan 35° = 12 ÷ x, so x × tan 35° = 12.
3. Divide both sides by tan 35°: x = 12 ÷ tan 35°.
4. On the calculator, to 4 decimal places, tan 35° = 0.7002.
5. Divide, to 3 decimal places: 12 ÷ 0.7002 = 17.138.
6. Round to 1 decimal place: x = 17.1 m.

Answer: To 1 decimal place, the viewer is 17.1 m from the tree.

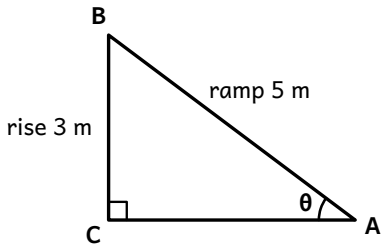
Keep the rule: when the unknown is on top of the ratio, multiply. When it is on the bottom, divide. In real work keep all the digits on your calculator and round only at the end. In the three examples we rounded the ratio to 4 places only to show each step. The final answers are the same.

SEE IT THREE WAYS

With things

A ramp is 5 m long and rises 3 m. For every metre of ramp, you rise 3 ÷ 5 = 0.6 m.

As a picture



In symbols

sin θ = opposite ÷ hypotenuse = 3 ÷ 5 = 0.6.

TRY IT

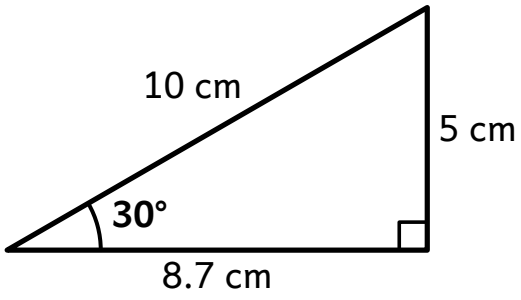
Measure a sine

You need: paper, a ruler, a protractor, a pencil.

1. Draw a flat line. At its left end use the protractor to draw a line at 30° above it, 10 cm long. This is the hypotenuse.
2. From the top end of the slanted line, draw a line straight down to the flat line. Measure it.
3. Divide the measured length by 10.

What you will see: The vertical line is about 5 cm, so the sine is 5 ÷ 10 = 0.5. Your calculator says sin 30° = 0.5. The paper and the calculator agree.

If you cannot try it, this is what happens.



A WRONG IDEA MANY PEOPLE HAVE

~~**On my calculator sin 30° is −0.988, so the calculator has a fault.**~~

Why it feels right: A calculator is exact, so a strange answer must be a broken button.

The test that breaks it: Look for DEG or D on the screen. Some calculators start in radians. In radians mode, sin 30 means 30 radians, and that gives −0.988.

What is true: Set the calculator to degrees. Then sin 30° = 0.5, which matches the drawing.

A WRONG IDEA MANY PEOPLE HAVE

~~**If sin 30° = 0.5, then sin 60° = 1, because 60° is twice 30°.**~~

Why it feels right: Doubling the angle looks like it should double the answer.

The test that breaks it: Draw a 60° angle with a hypotenuse of 10 cm. The upright side is about 8.7 cm, so sin 60° is about 0.87, not 1.

What is true: The ratios do not grow in step with the angle. Use the calculator for each angle.

KEY WORDS

sine
sin θ = opposite ÷ hypotenuse.

cosine
cos θ = adjacent ÷ hypotenuse.

tangent
tan θ = opposite ÷ adjacent.

BACK TO YOUR GUESS

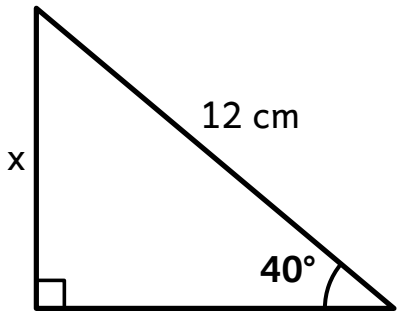
The answer is C: About 4.8 m, because the ladder is nearly upright.

The ladder reaches about 4.8 m. It cannot reach 5 m, because the ladder is the hypotenuse and the height is a shorter side. A percentage of the angle is not a length. You find the height with 5 × sin 75°, and this chapter shows how.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. You know the hypotenuse, 12 cm, and the angle, 40°. You want the opposite side x. Which ratio do you use?



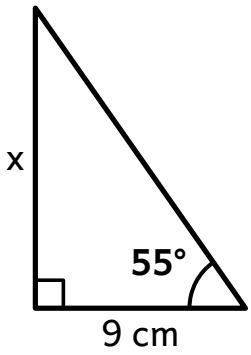
   A cos, because it joins adjacent and hypotenuse

   B sin, because it joins opposite and hypotenuse

   C tan, because it joins opposite and adjacent

   Answer at the back of the book.

2. You know the adjacent side, 9 cm, and the angle, 55°. You want the opposite side x. Which ratio do you use?



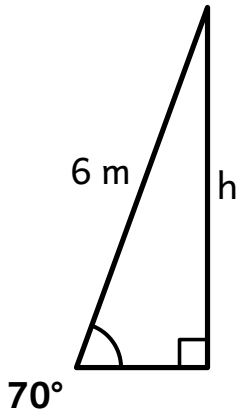
A sin, because it joins opposite and hypotenuse

B cos, because it joins adjacent and hypotenuse

C tan, because it joins opposite and adjacent

Answer at the back of the book.

3. A ladder 6 m long leans on a wall at 70° to the ground. How high up the wall does it reach, to 1 decimal place?



A 5.6 m

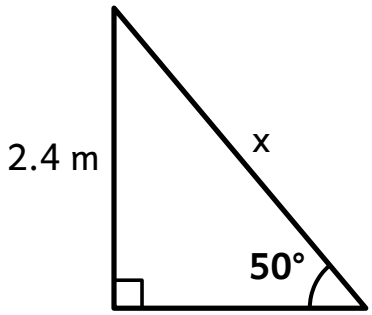
B 2.1 m

C 16.5 m

D 4.6 m

Answer at the back of the book.

4. A slide is 2.4 m high at its top end. It slopes at 50° to the ground. How long is the slide, to 1 decimal place?



A 1.8 m

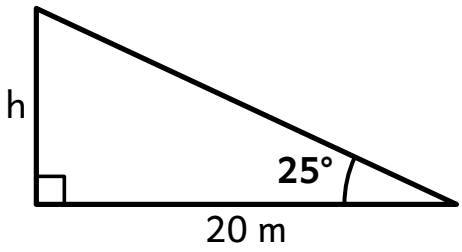
B 3.7 m

C 2.0 m

D 3.1 m

Answer at the back of the book.

5. You stand 20 m from the foot of a pole on level ground. You see the top at 25° above the level of your eye. How tall is the pole above your eye level, to 1 decimal place?



A 42.9 m

B 9.3 m

C 8.5 m

D 18.1 m

Answer at the back of the book.

6. **From an earlier chapter**

   A small triangle has opposite 4 cm and adjacent 5 cm for θ. A bigger one has the same θ and adjacent 20 cm. How long is its opposite side?

   A 19 cm

   B 25 cm

   C 16 cm

   D 80 cm

   Answer at the back of the book.

7. A ladder 8 m long leans on a wall at 72° to the ground. How many metres up the wall does it reach, to 1 decimal place?

   Answer at the back of the book.

8. A ramp rises 1.5 m and makes an angle of 12° with the ground. How many metres long is the ramp, to 1 decimal place?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

SOH CAH TOA: sin = O ÷ H, cos = A ÷ H, tan = O ÷ A. Label O, A, H. Pick the ratio with your two sides. Unknown on top: multiply. Unknown below: divide.

Draw: A right angled triangle with θ, the sides marked O, A and H, and the words SOH CAH TOA under it.

M4.3.5 CHAPTER 5

Finding angles

> **A ramp rises 1 m while it goes 4 m along the ground. What angle does it make with the ground?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** About 14°, found with the inverse tangent of 1 ÷ 4.
- ☐ **B** 25°, because 1 ÷ 4 = 0.25, and the ratio is the angle.
- ☐ **C** 45°, because the ramp goes up as far as it goes along.

IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Ratio and proportion, M3.4.1. If dividing one length by another to compare them looks new, read this first.
- Level 3, Letters for numbers and simple equations, M3.8.5. If solving 3x = 12 looks new, read this first.

BY THE END YOU CAN

- Use inverse sin, cos or tan to find an angle
- Find the angle of a slope from its height and length

In the last chapter the angle was given and a side was missing. Now it is the other way round. You know two sides and you want the angle.

You already know sin 30° = 0.5. Turn it round. Which angle has a sine of 0.5? The answer is 30°. The key that does this is the **inverse sine**, written $\sin^{-1}$. On most calculators you press shift, then sin.

Angle	sin	cos	tan
15°	0.26	0.97	0.27
30°	0.5	0.87	0.58
45°	0.71	0.71	1
60°	0.87	0.5	1.73
75°	0.97	0.26	3.73

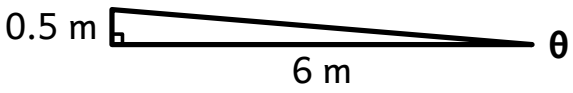
Reading the table from the right gives the inverse: which angle has sin 0.5? The row 30°.

Look at the table. As the angle grows, sin grows from 0 up to 1. A sine of 0.5 sits at 30°, and a sine of 0.87 at 60°. The inverse keys do the same search for any number, not only the ones in a small table.

Take care with the name. $\sin^{-1}$ does not mean 1 ÷ sin. The small −1 says "go backwards". You find an angle from a ratio, and the answer is an angle in degrees. Check that the calculator is in degrees mode.

The three inverse keys follow the same rule as before. Choose the ratio from the two sides you know. Divide them. Then press the inverse key for that ratio.

A sine or cosine in a right angled triangle is always between 0 and 1, because the hypotenuse is the longest side. If your ratio is above 1, the sides are the wrong way round. The calculator shows an error for $\sin^{-1}$ or $\cos^{-1}$ of a number above 1. A tangent can be any number above 0.



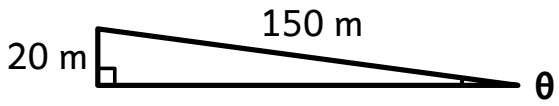
WORKED EXAMPLE

A ramp rises 0.5 m over a flat distance of 6 m. What angle does it make with the ground? Give the answer to 1 decimal place.

1. The rise 0.5 m is opposite θ. The flat 6 m is adjacent. We have O and A, so use tan.
2. tan θ = 0.5 ÷ 6.
3. Use the inverse tan key on 0.5 ÷ 6, without rounding the division first.
4. To 4 decimal places, θ = 4.7636.
5. Round to 1 decimal place: θ = 4.8°.

Answer: To 1 decimal place, the angle is 4.8°.

Now a road. A road is 150 m long and climbs 20 m. The road is the hypotenuse, so the angle uses sine.



WORKED EXAMPLE

A road is 150 m long and climbs 20 m. What angle does the road make with the flat? Give the answer to 1 decimal place.

1. The climb 20 m is opposite θ. The road, 150 m, is the hypotenuse. We have O and H, so use sin.
2. To 4 decimal places, sin θ = 20 ÷ 150 = 0.1333.
3. Use the inverse sin key on 20 ÷ 150.
4. To 4 decimal places, θ = 7.6623.
5. Round to 1 decimal place: θ = 7.7°.

Answer: To 1 decimal place, the angle is 7.7°.

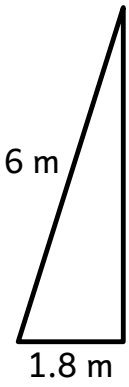
Last, a ladder. Its foot is 1.8 m from the wall and it is 6 m long. What angle does it make with the ground? Build the picture in three moves.

DRAW IT FIRST

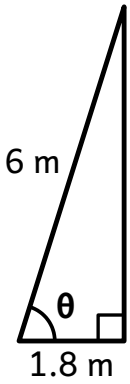
Find the angle of a ladder



1 Draw the ladder, the ground and the wall as a right angled triangle.



2 Write the known lengths: the foot distance 1.8 m and the ladder 6 m.



3 Mark θ where the ladder meets the ground.

WORKED EXAMPLE

A ladder 6 m long has its foot 1.8 m from a wall. What angle does it make with the ground? Give the answer to 1 decimal place.

1. The foot distance 1.8 m is adjacent to θ. The ladder, 6 m, is the hypotenuse. We have A and H, so use cos.
2. cos θ = 1.8 ÷ 6 = 0.3.
3. Use the inverse cos key on 0.3.
4. To 4 decimal places, θ = 72.5424.
5. Round to 1 decimal place: θ = 72.5°.

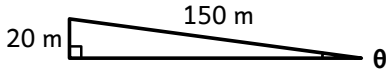
Answer: To 1 decimal place, the angle is 72.5°.

SEE IT THREE WAYS

With things

A road climbs 20 m for every 150 m you walk along it.

As a picture



In symbols

$\sin\theta = 20 \div 150$, so $\theta = \sin^{-1}(20 \div 150)$, which is 7.7° to 1 decimal place.

TRY IT

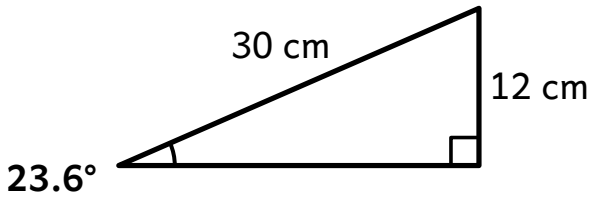
Find the angle of a ruler

You need: a ruler 30 cm long, a pile of books, a protractor.

1. Make a pile of books 12 cm high. Lean the ruler from the floor to the top of the pile.
2. The ruler is 30 cm long and the pile is 12 cm high. Work out $\sin^{-1}(12 \div 30)$.
3. Now measure the angle between the ruler and the floor with the protractor.

What you will see: The inverse sine gives 23.6°, and the protractor should show about the same.

If you cannot try it, this is what happens.



A WRONG IDEA MANY PEOPLE HAVE

~~The ratio is the angle: tan θ = 0.25 means θ = 25°.~~

Why it feels right: The digits 25 match, and a calculator shows a number, so the number seems to be the answer.

The test that breaks it: Work out tan 25° on the calculator. It gives 0.466, not 0.25. The angle that has tan 0.25 is 14.04°.

What is true: A ratio is a number about the shape, and an angle is a turn. Use the inverse key to change one into the other.

A WRONG IDEA MANY PEOPLE HAVE

~~$\sin^{-1} 0.5$ means $1 \div \sin 0.5$.~~

Why it feels right: In algebra a small −1 above a number means the reciprocal: $2^{-1} = 1 \div 2$.

The test that breaks it: We know $\sin 30° = 0.5$. If $\sin^{-1} 0.5$ were $1 \div \sin 0.5$, the answer would not be 30. The inverse key gives 30°, so it goes backwards, it does not divide.

What is true: On a function key, $^{-1}$ means the inverse function. $\sin^{-1}$ undoes sin.

KEY WORDS

inverse sine
$\sin^{-1}$: the angle whose sine is a given ratio. Likewise $\cos^{-1}$ and $\tan^{-1}$.

BACK TO YOUR GUESS

The answer is A: About 14°, found with the inverse tangent of $1 \div 4$.

The angle is about 14°. The ratio 0.25 is not the angle. It is the tangent of the angle, and the inverse tan key on a calculator turns it back into 14.04°. A ramp at 45° would rise 4 m over 4 m, not 1 m.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. A right angled triangle has opposite 5 cm and hypotenuse 8 cm for the angle θ. Which sum gives θ?

   A θ = inverse sin of 5 ÷ 8

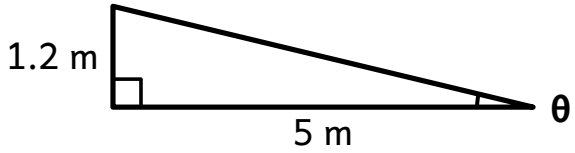
   B θ = inverse cos of 5 ÷ 8

   C θ = inverse tan of 5 ÷ 8

   D θ = 5 ÷ 8

   Answer at the back of the book.

2. A ramp rises 1.2 m over a flat distance of 5 m. What angle does it make with the ground, to 1 decimal place?



   A 76.5°

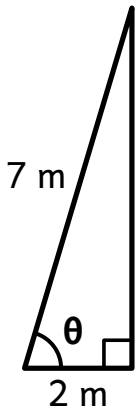
   B 13.5°

   C 13.9°

   D 0.24°

   Answer at the back of the book.

3. A ladder 7 m long has its foot 2 m from a wall. What angle does it make with the ground, to 1 decimal place?



A 16.6°

B 73.4°

C 15.9°

D 28.6°

Answer at the back of the book.

4. On a calculator, $\sin^{-1}$ of a number gives an error. The number came from dividing two sides of a right angled triangle. What is the most likely reason?

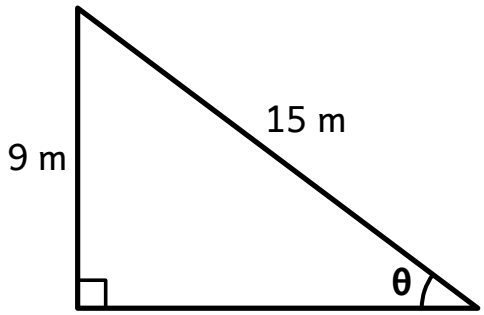
A The sides were divided the wrong way round, so the ratio was more than one.

B The calculator is broken.

C The angle is bigger than 90°.

Answer at the back of the book.

5. A steep path is 15 m long and climbs 9 m. What angle does it make with the flat, to 1 decimal place?



A 53.1°

B 31.0°

C 36.9°

D 0.6°

Answer at the back of the book.

6. **From an earlier chapter**

A ramp 8 m long slopes at 15° to the ground. How high does it rise, to 1 decimal place?

A 7.7 m

B 30.9 m

C 5.2 m

D 2.1 m

Answer at the back of the book.

7. A ramp rises 2 m over a flat distance of 9 m. What is the angle with the ground, in degrees, to 1 decimal place?

Answer at the back of the book.

8. A ladder 9 m long has its foot 2.4 m from a wall. What angle does it make with the ground, in degrees, to 1 decimal place?

Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Two sides known, angle wanted. Pick sin, cos or tan. Divide the sides, then press the inverse key. Use degrees mode.

Draw: A right angled triangle with two sides labelled, θ marked, and θ = $\tan^{-1}$(opposite ÷ adjacent) under it.

M4.3.6 CHAPTER 6

Exact values

> **On a calculator sin 45° shows 0.7071067812 and goes on. Is there a neat exact way to write it?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** No. The decimal never ends, so it cannot be written exactly.
- ☐ **B** Yes. It is exactly 0.7071, the calculator is only adding extra digits.
- ☐ **C** Yes. It is 1 ÷ √2, and half a square shows why.

IF THIS FEELS HARD, READ THIS FIRST

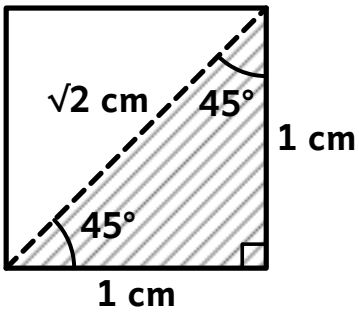
- Level 4, Powers, roots, surds and standard form, M4.1.7. If simplifying √2 or √3 looks new, read this first.
- Level 3, Angles, triangles and symmetry, M3.5.3. If equilateral triangles and their angles look new, read this first.

BY THE END YOU CAN

- Recall sin, cos and tan of 30°, 45° and 60°
- Work out those values from two special triangles

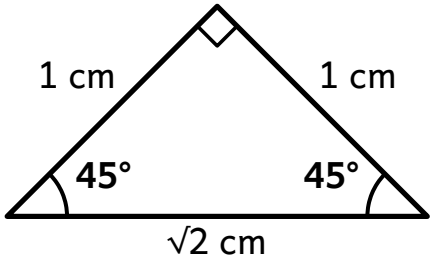
A calculator gives sin 60° as 0.8660254... The digits never end and never repeat, so no decimal is exact. But for 30°, 45° and 60° we can write the **exact value**, as a fraction or with a square root. We get them from two triangles that take a minute to draw.

First triangle. Draw a square with sides of 1 cm. Cut it along a diagonal. You get two equal right angled triangles.



The diagonal is √2 cm because $1^2 + 1^2 = 2$.

The diagonal cuts each right angle in half. So each triangle has angles 45°, 45° and 90°. Its sides are 1 cm, 1 cm and √2 cm.



WORKED EXAMPLE

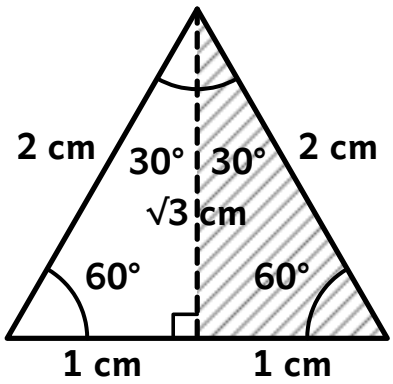
Use half a square with sides 1 cm to find sin 45°, cos 45° and tan 45° exactly.

1. Draw a square with sides 1 cm and cut it along a diagonal.
2. Each half has a right angle, two angles of 45° and two sides of 1 cm.
3. The hypotenuse by Pythagoras: $c^2 = 1^2 + 1^2 = 2$, so $c = \sqrt{2}$ cm.
4. For a 45° angle the opposite side is 1 and the hypotenuse is √2, so sin 45° = 1 ÷ √2.
5. The adjacent side is 1, so cos 45° = 1 ÷ √2 and tan 45° = 1 ÷ 1 = 1.

Answer: sin 45° = 1 ÷ √2, cos 45° = 1 ÷ √2 and tan 45° = 1.

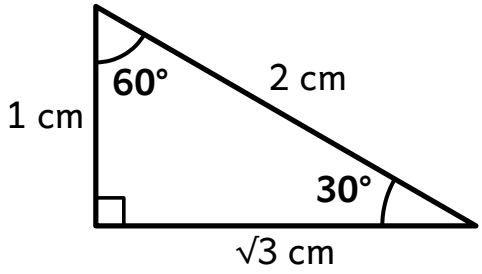
You may meet √2 ÷ 2 instead of 1 ÷ √2. They are the same number. Multiply the top and the bottom of 1 ÷ √2 by √2, and you get √2 ÷ 2. Both are 0.7071... Many books prefer √2 ÷ 2 because it has no root on the bottom.

Second triangle. Draw an equilateral triangle with sides 2 cm. Cut it down the middle from the top corner to the middle of the base.



The cut line is √3 cm because $2^2 - 1^2 = 3$.

The cut makes two right angled triangles. The top angle of 60° is split into two 30° angles. Each triangle has angles 30°, 60° and 90°, and sides 1 cm, √3 cm and 2 cm.



WORKED EXAMPLE

Use half an equilateral triangle with sides 2 cm to find sin 30°, cos 30° and tan 30° exactly.

1. Draw an equilateral triangle with sides 2 cm and cut it in half down the middle.
2. Each half has a right angle and angles of 30° and 60°. Its hypotenuse is 2 cm and its short side is 1 cm.
3. The third side by Pythagoras: $h^2 = 2^2 - 1^2 = 3$, so $h = \sqrt{3}$ cm.
4. For the 30° angle the opposite side is 1 and the hypotenuse is 2, so $\sin 30° = 1/2$.
5. The adjacent side is $\sqrt{3}$, so $\cos 30° = \sqrt{3} \div 2$ and $\tan 30° = 1 \div \sqrt{3}$.

Answer: $\sin 30° = 1/2$, $\cos 30° = \sqrt{3} \div 2$ and $\tan 30° = 1 \div \sqrt{3}$.

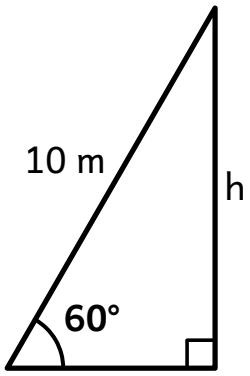
The same triangle gives the 60° values. The 60° angle is at the other end, so opposite and adjacent swap. Then $\sin 60° = \sqrt{3} \div 2$, $\cos 60° = 1/2$ and $\tan 60° = \sqrt{3} \div 1 = \sqrt{3}$.

Angle	sin	cos	tan
30°	1/2	√3/2	1/√3
45°	1/√2	1/√2	1
60°	√3/2	1/2	√3

Learn this table, or learn how to draw the two triangles and read it off.

In the table a slash is a division. So $1/\sqrt{3}$ means $1 \div \sqrt{3}$, and $\sqrt{3}/2$ means $\sqrt{3} \div 2$. The two ways of writing say the same thing.

Exact values keep an answer exact. A wire 10 m long holds a pole upright. It makes an angle of 60° with the ground. How high up the pole is it fixed? Leave the answer with a root first, then round it.



WORKED EXAMPLE

A wire 10 m long holds a pole upright and makes an angle of 60° with the ground. How high up the pole is it fixed? Give it exactly, then to 1 decimal place.

1. The height is opposite the 60° angle. The wire is the hypotenuse. We have O and H, so use sin.
2. sin 60° = √3 ÷ 2, so h = 10 × √3 ÷ 2.
3. Work out 10 ÷ 2 = 5.
4. So h = 5 × √3 = 5√3 m, exactly.
5. On the calculator, 5 × 1.7321 = 8.6605.
6. To 1 decimal place, h = 8.7 m.

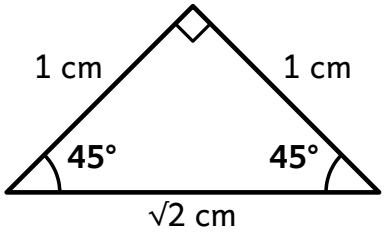
Answer: The wire is fixed exactly 5√3 m up the pole, which is 8.7 m to 1 decimal place.

SEE IT THREE WAYS

With things

Fold a square sheet of paper corner to corner. The fold is longer than a side, and the two halves are the same.

As a picture



In symbols

sin 45° = 1 ÷ √2, which is about 0.7071.

TRY IT

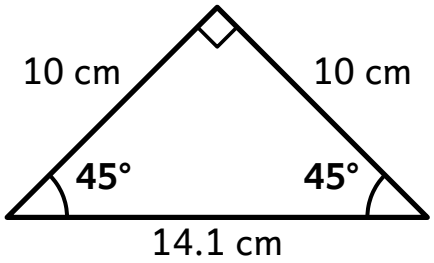
Fold a square

You need: a square sheet of paper, a ruler.

1. Cut a square with sides of 10 cm. Fold it corner to corner.
2. Measure the fold, which is the hypotenuse of each half.
3. Divide a side, 10 cm, by the fold.

What you will see: The fold is about 14.1 cm. The division 10 ÷ 14.1 is about 0.71. This is sin 45° = 1 ÷ √2, about 0.7071, as in the table.

If you cannot try it, this is what happens.



A WRONG IDEA MANY PEOPLE HAVE

~~cos 30° is 1/2, because sin 30° is 1/2 and the two go together.~~

Why it feels right: The numbers 1/2 and √3 ÷ 2 appear together in the table, and the two are often swapped.

The test that breaks it: Draw the 30° triangle. The side next to the 30° angle is the long leg, √3 cm, not the short one. So cos 30° = √3 ÷ 2, about 0.87. The calculator agrees.

What is true: sin 30° = 1/2 and cos 30° = √3 ÷ 2. Cos 60° is the one that equals 1/2. The sine and the cosine of 30° and 60° swap.

A WRONG IDEA MANY PEOPLE HAVE

~~1 ÷ √2 and √2 ÷ 2 are two different numbers.~~

Why it feels right: They look different, and one has a root on the bottom.

The test that breaks it: Multiply the top and the bottom of 1 ÷ √2 by √2. The top becomes 1 × √2 = √2. The bottom becomes √2 × √2 = 2. So 1 ÷ √2 = √2 ÷ 2.

What is true: They are the same number, 0.7071... Writing a root off the bottom is a matter of habit, not of value.

KEY WORDS

exact value
a value written with a fraction or a root, not rounded to decimals.

BACK TO YOUR GUESS

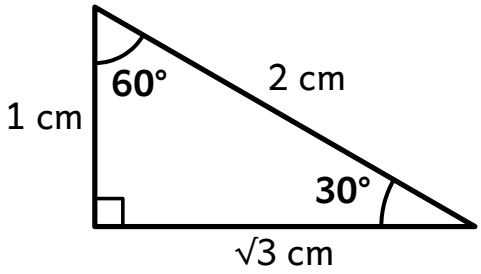
The answer is C: Yes. It is 1 ÷ √2, and half a square shows why.

It is exactly 1 ÷ √2. A decimal that never ends can still be written exactly with a root. Half a square gives sin 45°, and half an equilateral triangle gives the values for 30° and 60°. In this chapter you will see where each comes from.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. What is sin 30° exactly?



A √3 ÷ 2

B 1 ÷ √3

C 1/2

D 1

Answer at the back of the book.

2. What is tan 60° exactly?

A 1 ÷ √3

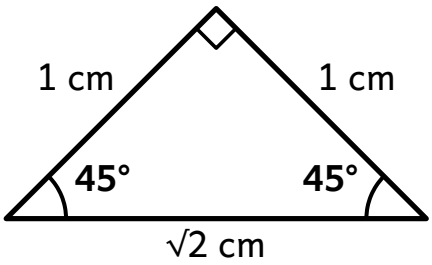
B √3

C √3 ÷ 2

D 1

Answer at the back of the book.

3. What is sin 45° exactly?



A 1/2

B √2

C 1

D 1 ÷ √2

Answer at the back of the book.

4. Which angle has a tangent of exactly 1?

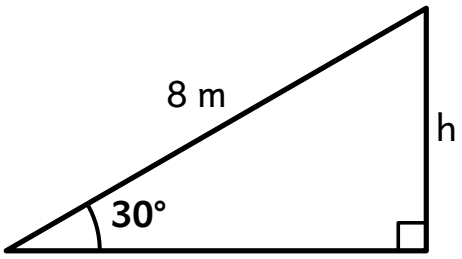
   A 30°

   B 45°

   C 60°

   Answer at the back of the book.

5. A rope 8 m long makes an angle of 30° with the ground. How high is its top end?



   A $4\sqrt{3}$ m

   B 16 m

   C 4 m

   D $8\sqrt{3}$ m

   Answer at the back of the book.

6. **From an earlier chapter**

   A ramp rises 0.9 m over a flat distance of 7.5 m. What angle does it make with the ground, to 1 decimal place?

   A 6.8°

   B 83.2°

   C 0.12°

   D 6.9°

   Answer at the back of the book.

7. A rope 14 m long makes an angle of 30° with the ground. How many metres high is its top end?

   Answer at the back of the book.

8. A roof slopes at 45° to the flat and rises 3.5 m. How many metres is the flat distance it covers?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Half a square gives 45°: sides 1, 1, √2. Half an equilateral triangle gives 30° and 60°: sides 1, √3, 2. Read sin, cos and tan from these.

Draw: The two triangles with their sides and angles, and a small table of sin, cos and tan for 30°, 45° and 60°.

M4.3.7 CHAPTER 7

The sine rule and the cosine rule

> **A triangle has no right angle. You know two of its angles and one side. How can you find the other sides?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** Cut it into right angled triangles, and the work gives two rules that work for any triangle.
- ☐ **B** You cannot. Sine, cosine and tangent only work with a right angle.
- ☐ **C** Guess the right angle, then use Pythagoras.

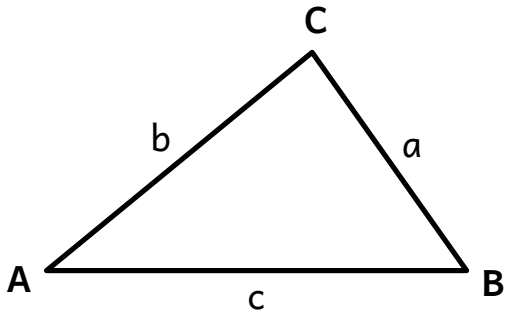
IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Angles, triangles and symmetry, M3.5.3. If the angles of a triangle adding to 180° look new, read this first.
- Level 3, Letters for numbers and simple equations, M3.8.4. If expanding 3(x + 4) looks new, read this first.

BY THE END YOU CAN

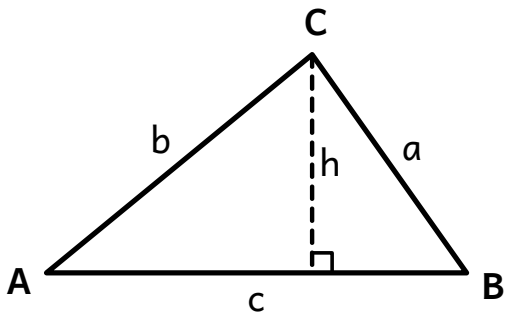
- Use the sine rule to find a side or an angle
- Use the cosine rule when you know two sides and the angle between

Name a triangle ABC. Use a capital letter for each angle, and the small letter for the side opposite it. So side a faces angle A, side b faces angle B and side c faces angle C.



Each side is named after the corner it faces.

Draw the height h from C down to the side c. It makes two right angled triangles. In the left one, sin A = h ÷ b, so h = b × sin A. In the right one, sin B = h ÷ a, so h = a × sin B.

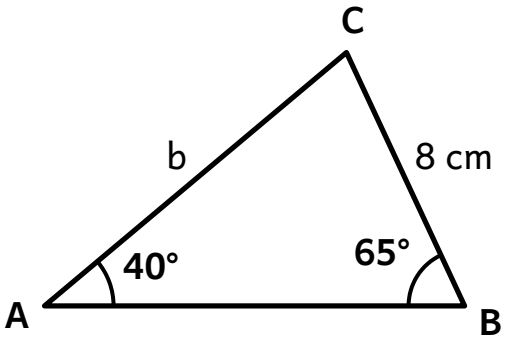


The height h is a side of two right angled triangles.

Both are h, so b × sin A = a × sin B. Divide both sides by sin A × sin B. On the left you are left with b ÷ sin B. On the right you are left with a ÷ sin A. So a ÷ sin A = b ÷ sin B. Cutting from another corner gives the same with c.

This is the **sine rule**: a ÷ sin A = b ÷ sin B = c ÷ sin C. Our height fell inside the triangle, because angles A and B were both acute. The rule is true for every triangle. A little later we see why it holds for an obtuse angle too.

Use it when you have a matching pair: an angle and the side opposite it. Then one more piece lets you find another pair. Here is a side.



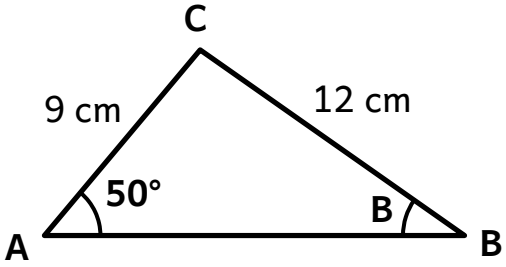
WORKED EXAMPLE

In triangle ABC, A = 40°, B = 65° and *a* = 8 cm. Find the side b, to 1 decimal place.

1. We know *a* and A, a matching pair, and B. We want b, opposite B. Use the sine rule.
2. *a* ÷ sin A = b ÷ sin B, so b = *a* × sin B ÷ sin A.
3. b = 8 × sin 65° ÷ sin 40°.
4. To 4 decimal places, sin 65° = 0.9063 and sin 40° = 0.6428.
5. To 4 decimal places, 8 × 0.9063 ÷ 0.6428 = 11.2794.
6. Round to 1 decimal place: b = 11.3 cm.

Answer: To 1 decimal place, b is 11.3 cm.

You can also find an angle. Turn the rule upside down: sin A ÷ *a* = sin B ÷ b. Then sin B = b × sin A ÷ *a*, and the inverse sin key gives B.



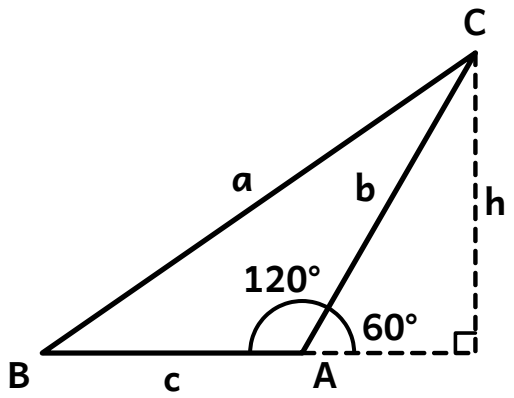
WORKED EXAMPLE

In triangle ABC, A = 50°, *a* = 12 cm and b = 9 cm. Find the angle B, to 1 decimal place.

1. We know *a* and A, a matching pair, and b. We want B, opposite b. Use the sine rule upside down.
2. sin B ÷ b = sin A ÷ *a*, so sin B = b × sin A ÷ *a*.
3. sin B = 9 × sin 50° ÷ 12.
4. To 4 decimal places, sin B = 9 × 0.7660 ÷ 12 = 0.5745.
5. Use the inverse sin key on 9 × sin 50° ÷ 12, with all the digits. To 3 decimal places, B = 35.067.
6. Round to 1 decimal place: B = 35.1°.
7. Here b is shorter than *a*, so B is smaller than A and must be acute. 35.1° is the only answer.

Answer: To 1 decimal place, B is 35.1°.

So far every angle was below 90°. A triangle can have an angle above 90°, called an obtuse angle. Then the height from C falls outside the triangle, on the line c extended beyond A.



The height falls outside. The angle outside at A is 180° − 120° = 60°.

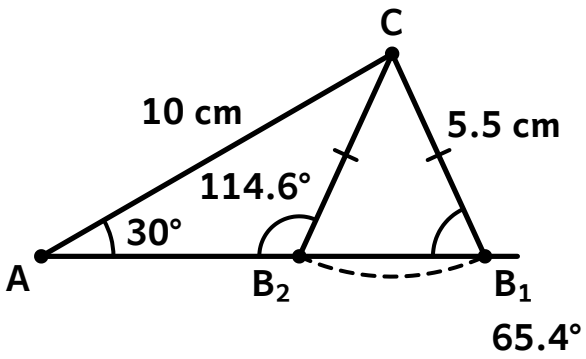
The right angled triangle outside has the angle 180° − A at A. In the picture that is 60°. Its height is h = b × sin 60°.

We want h = b × sin A to stay true for the obtuse angle A. So we choose sin A to mean sin(180° − A). It is a definition. We choose it so that the rules in this chapter keep working.

We make the same choice for the cosine: cos A = −cos(180° − A). Here is why. When A is obtuse, the foot of the height lies beyond A. So the far part of the base is c + b cos(180° − A). With our choice that is c − b cos A again, so the cosine rule keeps its form. Check on a calculator. sin 114.6° and sin 65.4° both give 0.9092. cos 100° gives −0.1736 and cos 80° gives 0.1736. A negative cosine for an angle above 90° is not a fault.

Take care. When you find an angle with the sine rule, there can be two answers. Two different angles have the same sine: an acute angle and 180° minus it. The calculator gives only the acute one.

Look at the picture. Both triangles have A = 30°, a = 5.5 cm and b = 10 cm. Swing the 5.5 cm side round C. It touches the base line in two places, B_1 and B_2. Both make real triangles: 30° + 65.4° and 30° + 114.6° are both under 180°.



The same three facts, two triangles. The tick marks CB_1 and CB_2 as equal. B is 65.4° at B_1 or 114.6° at B_2.

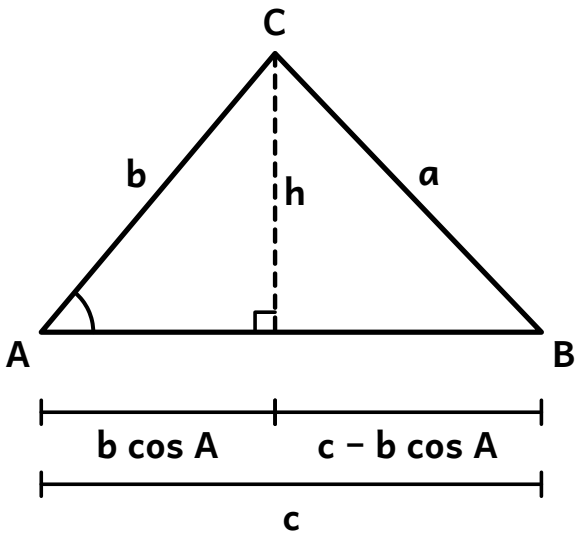
The two answers for B are 65.4° and 180° − 65.4° = 114.6°. Always test the second angle. Add it to the known angle: 30° + 114.6° = 144.6°, which is below 180°. So the second triangle exists.

There is at most one triangle in these cases. First, the angle you find is opposite the shorter of the two known sides. Then it must be acute. That was the case before: B faced b = 9 cm, shorter than a = 12 cm.

Second, the known angle is 90° or more. Then the side opposite it must be the longest side, or there is no triangle. Third, the known angle and the second angle add to 180° or more. If sin B works out above 1, there is no triangle at all.

The sine rule needs a matching pair. Sometimes you have no pair. You may know two sides and the angle between them, or all three sides. Then use the second rule.

Draw the height h from C onto side c again. It cuts c into two parts. The part next to A is b × cos A. The rest is c − b × cos A. The height is h = b × sin A.



The foot of the height cuts c into b cos A and c − b cos A.

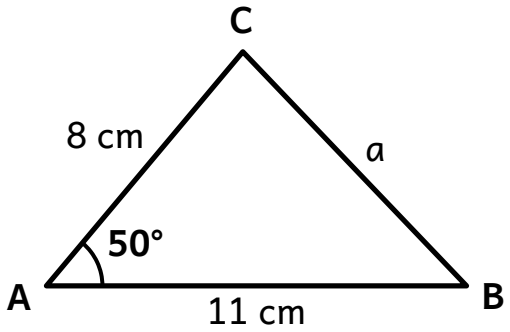
Now use Pythagoras in the right angled triangle on the right: $a^2 = h^2 + (c - b \cos A)^2$. Put in h = b sin A.

First expand the bracket. Multiply each part of (c − b cos A) by each part of the other bracket. Then $(c - b \cos A)(c - b \cos A) = c^2 - 2bc \cos A + b^2 \cos^2 A$.

So $a^2 = b^2 \sin^2 A + c^2 - 2bc \cos A + b^2 \cos^2 A$. Collect the two b^2 terms. They give $b^2 \times (\sin^2 A + \cos^2 A)$.

In the left triangle Pythagoras says $(b \sin A)^2 + (b \cos A)^2 = b^2$. So $\sin^2 A + \cos^2 A = 1$, and the sum is b^2. Therefore $a^2 = b^2 + c^2 - 2bc \cos A$.

This is the **cosine rule**: $a^2 = b^2 + c^2 - 2bc \cos A$. Our height fell inside, but the rule holds for every triangle. For an obtuse A the cosine is negative, so $-2bc \cos A$ adds to the sum. If A = 90°, then cos A = 0 and the rule is just Pythagoras. The extra part $-2bc \cos A$ corrects for the angle not being 90°.



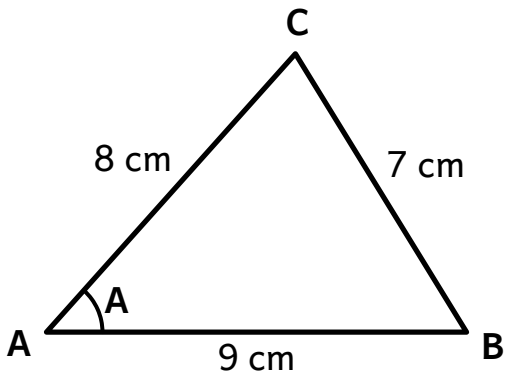
WORKED EXAMPLE

In triangle ABC, b = 8 cm, c = 11 cm and A = 50°. Find the side *a*, to 1 decimal place.

1. We know two sides, b = 8 and c = 11, and the angle A = 50° between them. We want *a*. Use the cosine rule.
2. $a^2 = b^2 + c^2 - 2 \times b \times c \times \cos A = 8^2 + 11^2 - 2 \times 8 \times 11 \times \cos 50°$.
3. Work out the squares and the product: $a^2 = 64 + 121 - 176 \times \cos 50°$.
4. To 4 decimal places, $\cos 50° = 0.6428$.
5. Then $a^2 = 185 - 176 \times 0.6428 = 71.8672$.
6. Take the square root, to 3 decimal places: $a = 8.477$.
7. Round to 1 decimal place: $a = 8.5$ cm.

Answer: To 1 decimal place, *a* is 8.5 cm.

The cosine rule can also find an angle when you know all three sides. Rearrange it: $\cos A = (b^2 + c^2 - a^2) \div (2 \times b \times c)$. Then use the inverse cos key.



WORKED EXAMPLE

A triangle has sides *a* = 7 cm, b = 8 cm and c = 9 cm. Find the angle A, opposite *a*, to 1 decimal place.

1. We know all three sides. We want A, opposite *a*. Use the cosine rule.
2. Rearrange: $\cos A = (b^2 + c^2 - a^2) \div (2 \times b \times c)$.
3. $\cos A = (8^2 + 9^2 - 7^2) \div (2 \times 8 \times 9)$.
4. The top is 64 + 81 − 49 = 96.
5. The bottom is 2 × 8 × 9 = 144.
6. To 4 decimal places, cos A = 96 ÷ 144 = 0.6667.
7. Use the inverse cos key on 96 ÷ 144. To 4 decimal places, A = 48.1897.
8. Round to 1 decimal place: A = 48.2°.

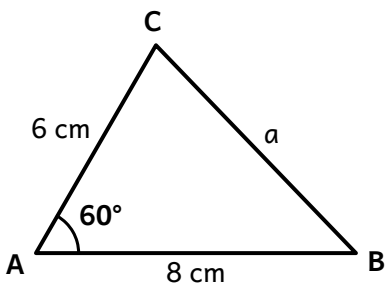
Answer: To 1 decimal place, A is 48.2°.

SEE IT THREE WAYS

With things

Two sticks joined at one end, 6 cm and 8 cm long, opened to a 60° angle. How far apart are the free ends?

As *a* picture



In symbols

$a^2 = 6^2 + 8^2 - 2 \times 6 \times 8 \times \cos 60° = 36 + 64 - 48 = 52$. So $a = \sqrt{52}$, which is 7.2 cm to 1 decimal place.

TRY IT

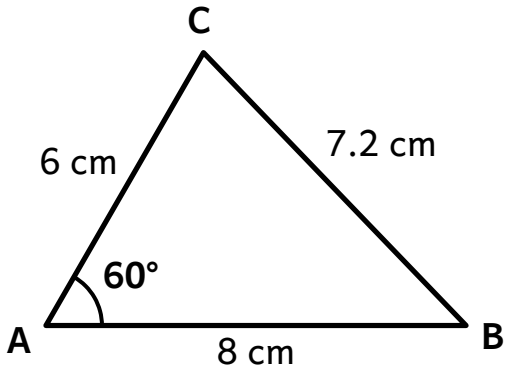
Two sticks and a string

You need: two sticks or pencils of 6 cm and 8 cm, a protractor, a ruler.

1. Lay the two sticks from one point, with a 60° angle between them. Use the protractor.
2. Measure the distance between the free ends.
3. Work out $6^2 + 8^2 - 2 \times 6 \times 8 \times \cos 60°$, then take the square root.

What you will see: The sum gives √52, which is 7.2 cm to 1 decimal place. Your measure should be close.

If you cannot try it, this is what happens.



A WRONG IDEA MANY PEOPLE HAVE

~~Use the sine rule or the cosine rule, whichever comes to mind first.~~

Why it feels right: Both rules have a side and an angle in them, so either seems to fit.

The test that breaks it: Try two sides and the angle between them. The sine rule has no matching pair, so you cannot start. With a pair, the sine rule needs just one more piece.

What is true: Pick the rule from what you know. A matching pair (an angle and its opposite side) means the sine rule. Two sides and the angle between, or three sides, means the cosine rule.

A WRONG IDEA MANY PEOPLE HAVE

~~The sine rule always gives exactly one angle.~~

Why it feels right: The calculator shows one answer, and one answer looks final.

The test that breaks it: Take A = 30°, a = 5.5 cm and b = 10 cm. sin B = 0.909, so the calculator gives 65.4°. But 114.6° has the same sine, and a triangle with 30° and 114.6° is fine.

What is true: Sine gives two angles, B and 180° − B. The second fits only when the known angle plus 180° − B is under 180°. That needs B opposite the longer of the two known sides. If sin B works out above 1, no triangle has those measures.

KEY WORDS

sine rule
$a \div \sin A = b \div \sin B = c \div \sin C$, true for every triangle.

cosine rule
$a^2 = b^2 + c^2 - 2bc \cos A$, true for every triangle.

BACK TO YOUR GUESS

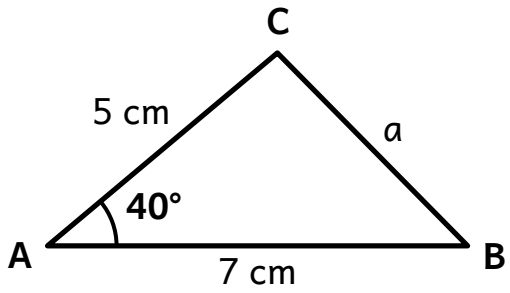
The answer is A: Cut it into right angled triangles, and the work gives two rules that work for any triangle.

Cutting a triangle with a height makes two right angled triangles. That gives the sine rule and the cosine rule, which work in every triangle. This chapter builds both rules from the height, then uses them.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. You know two sides of a triangle and the angle between them. Which rule finds the third side?



   A The sine rule

   B The cosine rule

   C Pythagoras

   Answer at the back of the book.

2. You know all three sides of a triangle and want one angle. Which rule do you use?

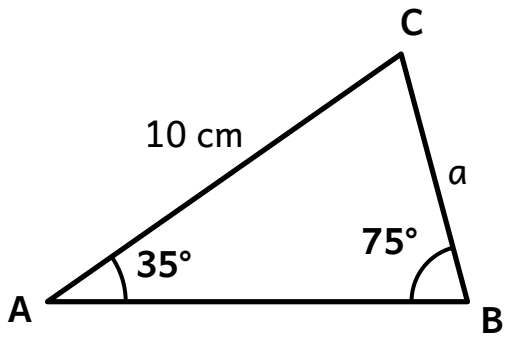
   A The cosine rule, rearranged to give cos A

   B The sine rule, because it has sines in it

   C Pythagoras, then sin

   Answer at the back of the book.

3. In triangle ABC, A = 35°, B = 75° and b = 10 cm. Find *a*, to 1 decimal place.



A 16.8 cm

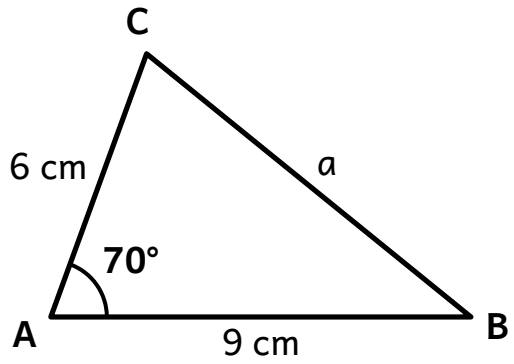
B 5.7 cm

C 5.9 cm

D 11.0 cm

Answer at the back of the book.

4. In triangle ABC, b = 6 cm, c = 9 cm and A = 70°. Find *a*, to 1 decimal place.



A 3.1 cm

B 12.4 cm

C 80.1 cm

D 8.9 cm

Answer at the back of the book.

5. In triangle ABC, A = 40°, a = 15 cm and b = 11 cm. Find the angle B, to 1 decimal place. Here a is longer than b, so B is acute.

   A 61.2°

   B 28.1°

   C 61.9°

   D 0.47°

   Answer at the back of the book.

6. A triangle has A = 40°, a = 8 cm and b = 11 cm. The sine rule gives B = 62.1°. What other value can B have?

   A 117.9°

   B 77.9°

   C 27.9°

   D There is no other value.

   Answer at the back of the book.

7. **From an earlier chapter**

   What is cos 60° exactly?

   A √3 ÷ 2

   B 1 ÷ √3

   C 1/2

   D 1

   Answer at the back of the book.

8. In triangle ABC, A = 50°, B = 70° and a = 12 cm. How many centimetres is b, to 1 decimal place?

   Answer at the back of the book.

9. In triangle ABC, b = 5 cm, c = 9 cm and A = 30°. How many centimetres is a, to 1 decimal place?

   Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Sine rule: $a \div \sin A = b \div \sin B$. Use it with a matching pair.
Cosine rule: $a^2 = b^2 + c^2 - 2bc \cos A$. Use it with two sides and the angle between, or three sides.

Draw: A triangle ABC with its height h dashed, and the two rules written under it.

M4.3.8 CHAPTER 8

Area of any triangle, and bearings

> **A walker leaves camp A and walks to B on a bearing of 065°. What bearing takes her back to A?**

YOUR GUESS

Pick one before you read on. It is fine to be wrong.

- ☐ **A** 115°, because 180° − 65° = 115°.
- ☐ **B** 245°, because turning round adds a half turn: 65° + 180° = 245°.
- ☐ **C** 295°, because 360° − 65° = 295°.

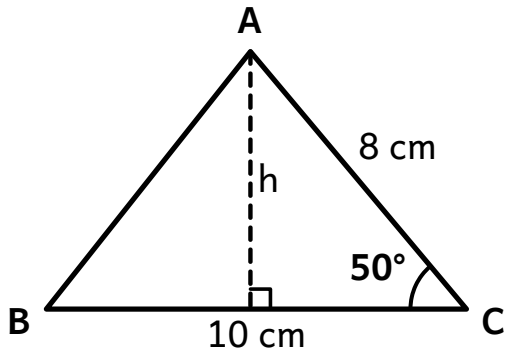
IF THIS FEELS HARD, READ THIS FIRST

- Level 3, Perimeter, area and volume, M3.6.4. If area = 1/2 × base × height looks new, read this first.
- Level 3, Angles, triangles and symmetry, M3.5.2. If angles between parallel lines look new, read this first.

BY THE END YOU CAN

- Find the area of a triangle as 1/2 ab sin C
- Give a direction as a three figure bearing, like 065°
- Solve a journey in two parts with bearings and the cosine rule

You know that the area of a triangle is $^1/_2$ × base × height. The height is not always given. Sometimes you know two sides and the angle between them. Then trigonometry finds the height for you.



Take a as the base. The height h stands on it.

In the right angled triangle at the right, $\sin C = h \div b$, so $h = b \times \sin C$. Then area = $\frac{1}{2} \times a \times h = \frac{1}{2} \times a \times b \times \sin C$. This is the **area rule**: area = $\frac{1}{2} ab \sin C$, where C is the angle between the sides a and b.

The angle must be the one between the two sides you use. Our height fell inside the triangle. If it falls outside, the sine of an obtuse angle gives the same height. The last chapter showed why, so the rule still holds.

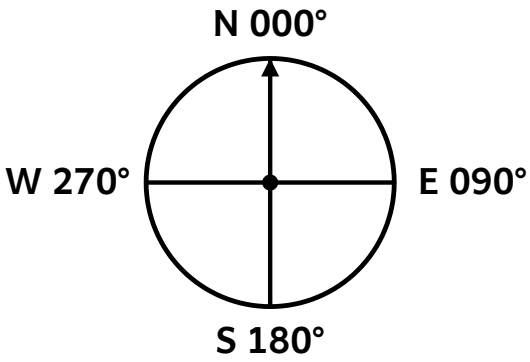
WORKED EXAMPLE

A triangle has sides a = 10 cm and b = 8 cm, and the angle between them is C = 50°. Find its area, to 1 decimal place.

1. The angle C = 50° is between the sides a = 10 and b = 8. Use area = $\frac{1}{2} \times a \times b \times \sin C$.
2. To 4 decimal places, $\sin 50° = 0.7660$.
3. Area = $0.5 \times 10 \times 8 \times 0.7660 = 30.64$.
4. Round to 1 decimal place: area = 30.6 cm^2.

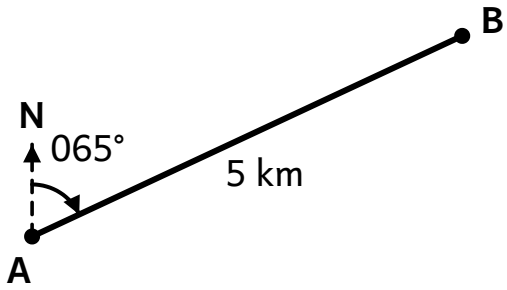
Answer: To 1 decimal place, the area is 30.6 cm^2.

Now bearings. A **bearing** gives a direction as an angle measured clockwise from north. It is always written with three figures, so 65° is written 065°. North is 000°, east is 090°, south is 180° and west is 270°.

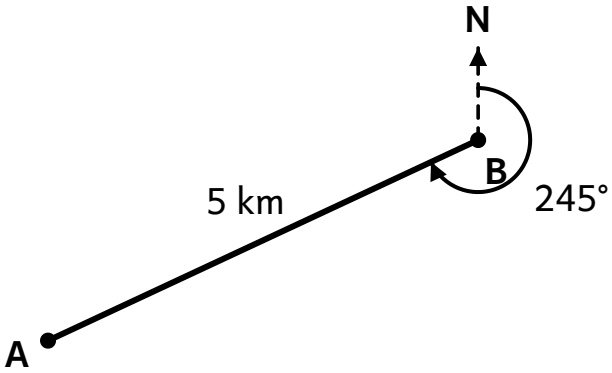


Bearings turn clockwise, the way a clock hand turns.

To draw a bearing, stand at the start. Draw a line north. Turn clockwise by the angle. Here is 065° from A to B, 5 km.



Going back from B to A is the opposite direction, a half turn of 180°. The north lines at A and B are parallel, so the bearing at B is 065° + 180° = 245°. If the sum is 360° or more, take 360° away. For a bearing of 250°, 250° + 180° = 430°. Take 360° away to get 070°. That is the same as taking 180° away, which is quicker for any bearing of 180° or more.



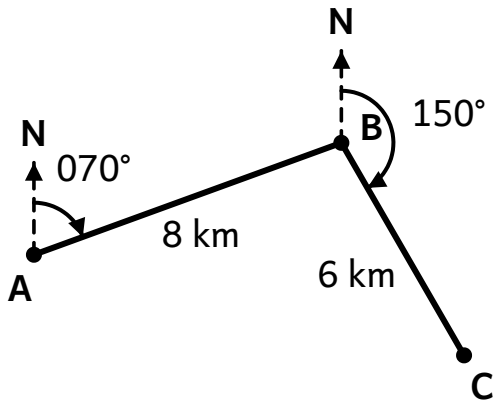
WORKED EXAMPLE

The bearing of B from A is 065°. Find the bearing of A from B.

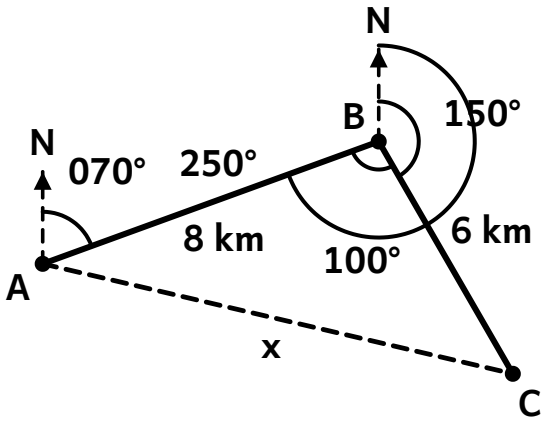
1. A to B is a bearing of 065°. Going back from B to A turns the direction through a half turn, 180°.
2. The north lines at A and B are parallel, so the half turn works at B too.
3. Add 180°: 65 + 180 = 245.
4. The result is below 360, so write it with three figures: 245°.

Answer: The bearing of A from B is 245°.

Now a journey in two parts. A boat leaves A and sails 8 km on a bearing of 070° to B. It turns and sails 6 km on a bearing of 150° to C. How far is C from A, and on what bearing?



First find the angle ABC inside the triangle. At B the way back to A is the bearing 070° + 180° = 250°. The way on to C is 150°. The angle between the two lines is 250° − 150° = 100°.



The angle inside the triangle at B is the difference of the two bearings: 250° − 150° = 100°.

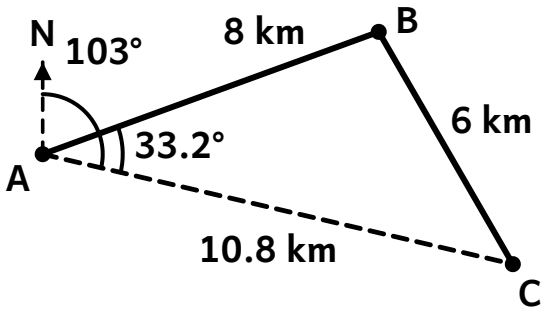
WORKED EXAMPLE

A boat sails 8 km on a bearing of 070° from A to B. Then it sails 6 km on a bearing of 150° to C. How far is C from A, to 1 decimal place?

1. Draw the journey. A is the start, B is the turn and C is the end.
2. At B the bearing of A is 70 + 180 = 250°. The bearing of C is 150°.
3. The angle ABC is the difference: 250 − 150 = 100°.
4. We know two sides, 8 and 6, and the angle between them. Use the cosine rule: $AC^2 = 8^2 + 6^2 - 2 \times 8 \times 6 \times \cos 100°$.
5. To 4 decimal places, cos 100° = −0.1736. The cosine of an obtuse angle is negative, as the last chapter showed.
6. Use all the digits of cos 100°. To 4 decimal places, $AC^2 = 116.6702$.
7. To 4 decimal places, AC = 10.8014.
8. Round to 1 decimal place: AC = 10.8 km.

Answer: To 1 decimal place, C is 10.8 km from A.

Now the bearing of C from A. Find the angle BAC with the sine rule. Then add it to 070°, because from A the point C lies clockwise of B.



The bearing of C from A is the bearing of B, 070°, plus the angle BAC, 33.2°.

WORKED EXAMPLE

Use the same journey. Find the bearing of C from A, to the nearest degree.

1. We want the angle BAC. BC = 6 is opposite it, and AC = 10.8014 is opposite the angle 100°. Use the sine rule.
2. sin A ÷ 6 = sin 100° ÷ 10.8014, so sin A = 6 × sin 100° ÷ 10.8014.
3. Use all the digits on your calculator. To 4 decimal places, sin A = 0.5470.
4. The inverse sin key gives, to 2 decimal places, A = 33.16.
5. BC is the shortest side, so A is acute and 33.16° is the only answer.
6. From A, the point B is on 070° and C is clockwise of B. The bearing of C is 70 + 33.16 = 103.16.
7. Round to the nearest degree: the bearing of C from A is 103°.

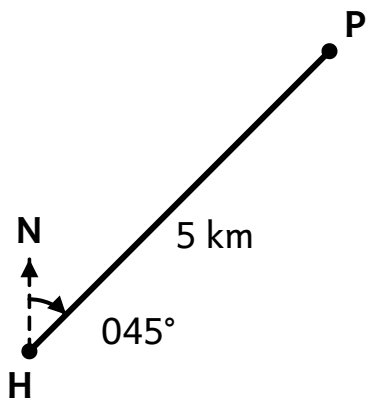
Answer: The bearing of C from A is 103°.

SEE IT THREE WAYS

With things

A path leaves a hut and runs north east. North east is half way between north and east.

As a picture



In symbols

North east = half of 090° = 045°.

TRY IT

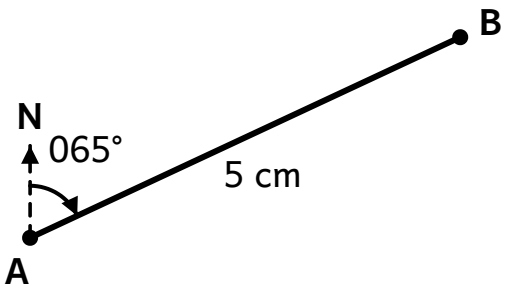
Draw a bearing

You need: paper, a ruler, a protractor.

1. Mark a point A. Draw a straight line up the page from A, 3 cm long. This is north.
2. Place the protractor at A. Measure 65° clockwise from the north line, and mark it.
3. Draw a line 5 cm long from A through the mark. Label the end B.

What you will see: The line AB points in the direction 065°. Now stand at B, draw a north line, and measure clockwise to A. You should find 245°.

If you cannot try it, this is what happens.



A WRONG IDEA MANY PEOPLE HAVE

~~Bearings turn anticlockwise, like angles in a maths class.~~

Why it feels right: Angles in a graph are measured anticlockwise from the right, so it seems natural to do the same with directions.

The test that breaks it: West is 270° on a bearing. If bearings turned anticlockwise, a quarter turn from north would be west, 090°. The compass picture shows east is 090°.

What is true: A bearing starts at north and turns clockwise, the way a clock hand does, and it has three figures.

A WRONG IDEA MANY PEOPLE HAVE

~~The area rule works with any angle in the triangle.~~

Why it feels right: There are three angles and sin C looks like a general part of the rule.

The test that breaks it: For sides 10 cm and 8 cm with 50° between them the area is 30.6 cm^2. Using the 51.6° angle at B, which is not between those sides, gives 31.4 cm^2. That is a different answer, and it is wrong.

What is true: The angle must be between the two sides in the formula. If you have sides a and b, use angle C. If you have b and c, use A.

KEY WORDS

area rule
area = $^1/_2$ ab sin C, with C the angle between sides a and b.

bearing
a direction given as an angle clockwise from north, written with three figures.

BACK TO YOUR GUESS

The answer is B: 245°, because turning round adds a half turn: 65° + 180° = 245°.

The bearing back is 245°. A bearing is the angle clockwise from north. Going back is a half turn, 180°, from the way you came, so you add 180°. This chapter explains bearings, and finds areas of triangles with no right angle.

CHECK YOURSELF

Try each one before you look. A wrong answer tells you where to go next.

1. A ship sails due west. What is its bearing?

   A 090°

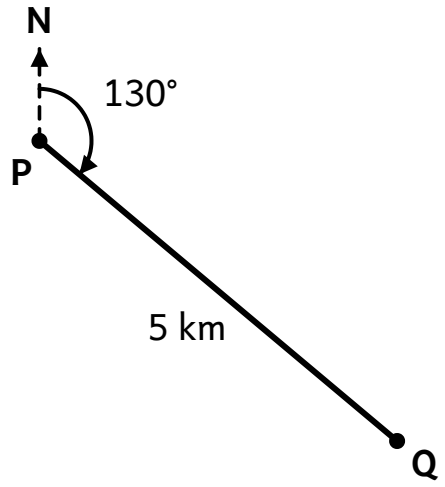
   B 270°

   C 180°

   D 000°

   Answer at the back of the book.

2. The bearing of Q from P is 130°. What is the bearing of P from Q?



   A 310°

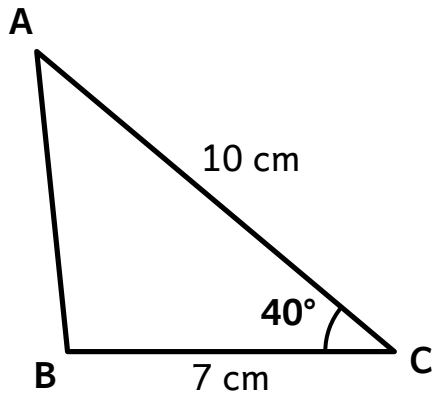
   B 050°

   C 230°

   D 130°

   Answer at the back of the book.

3. A triangle has sides 7 cm and 10 cm with an angle of 40° between them. What is its area, to 1 decimal place?



A 45.0 cm^2

B 26.8 cm^2

C 22.5 cm^2

D 35.0 cm^2

Answer at the back of the book.

4. A triangle has sides 8 cm and 7 cm, and its area is 20 cm^2. The angle C between the two sides is not shown. What can C be, to 1 decimal place?

A 45.6° or 134.4°

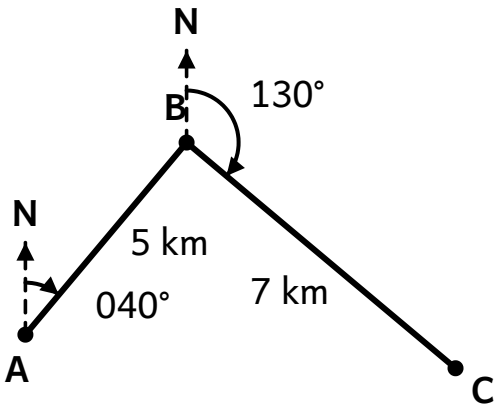
B 45.6° only

C 44.4° or 135.6°

D 0.71°

Answer at the back of the book.

5. A ship sails 5 km on a bearing of 040°. It then sails 7 km on a bearing of 130°. How far is it from where it started, to 1 decimal place?



A 12.0 km

B 10.9 km

C 4.5 km

D 8.6 km

Answer at the back of the book.

6. **From an earlier chapter**

In triangle ABC, a = 7 cm, b = 5 cm and c = 8 cm. Find the angle C, to 1 decimal place.

A 98.2°

B 0.14°

C 81.8°

Answer at the back of the book.

7. A triangle has sides 9 cm and 11 cm with an angle of 55° between them. How many square centimetres is its area, to 1 decimal place?

Answer at the back of the book.

8. The bearing of B from A is 250°. What is the bearing of A from B? Write the number.

Answer at the back of the book.

KEEP IT

Copy this into your exercise book.

Area = half × *a* × b × sin C, with C between *a* and b. A bearing is clockwise from north, in three figures. Back bearing: add or take away 180°.

Draw: A triangle with the angle C between sides *a* and b. Beside it, a north line with a bearing arc.

Key words, A to Z

Every key word in this book, what it means, and the chapter that teaches it.

adjacent
the side next to the angle θ that is not the hypotenuse. M4.3.3

area rule
area = $\frac{1}{2}$ ab sin C, with C the angle between sides a and b. M4.3.8

bearing
a direction given as an angle clockwise from north, written with three figures. M4.3.8

cosine
cos θ = adjacent ÷ hypotenuse. M4.3.4

cosine rule
$a^2 = b^2 + c^2 - 2bc \cos A$, true for every triangle. M4.3.7

exact value
a value written with a fraction or a root, not rounded to decimals. M4.3.6

hypotenuse
the longest side of a right angled triangle, opposite the right angle. M4.3.1

inverse sine
$\sin^{-1}$: the angle whose sine is a given ratio. Likewise $\cos^{-1}$ and $\tan^{-1}$. M4.3.5

opposite
the side across from the angle θ, not touching it. M4.3.3

Pythagoras' theorem
in a right angled triangle, $a^2 + b^2 = c^2$, with c the hypotenuse. M4.3.1

sine
sin θ = opposite ÷ hypotenuse. M4.3.4

sine rule
$a \div \sin A = b \div \sin B = c \div \sin C$, true for every triangle. M4.3.7

space diagonal
the line from one corner of a box to the opposite corner, through the inside. M4.3.2

tangent
tan θ = opposite ÷ adjacent. M4.3.4

Answers

M4.3.1. Pythagoras' theorem

1. **PR**

   The right angle is at Q. The side opposite it is PR, so PR is the hypotenuse. It is also the longest side.

   - **If you chose A:** PQ is a leg: it meets the right angle at Q. The hypotenuse does not touch the right angle. Go to M4.3.1.
   - **If you chose C:** QR is a leg too. It ends at the right angle. The hypotenuse is the side facing the right angle. Go to M4.3.1.
   - **If you chose D:** It cannot. The hypotenuse is always the side opposite the right angle, and it is always the longest side. Go to M4.3.1.

2. **15 cm**

   $c^2 = 9^2 + 12^2 = 81 + 144 = 225$, and $\sqrt{225} = 15$.

   - **If you chose A:** You added the legs, 9 + 12. The hypotenuse is shorter than the two legs added together. Square first, add, then take the root. Go to M4.3.1.
   - **If you chose B:** 225 is c^2, the square of the hypotenuse: 81 + 144. Take the square root to find c itself. Go to M4.3.1.
   - **If you chose C:** You took the shorter leg from the longer one. For a hypotenuse you add the squares, not subtract the lengths. Go to M4.3.1.

3. **15 cm**

   $a^2 = 17^2 - 8^2 = 289 - 64 = 225$, and $\sqrt{225} = 15$.

   - **If you chose B:** You added the squares, as for a hypotenuse. Here the hypotenuse is given, so subtract: $17^2 - 8^2$. Go to M4.3.1.
   - **If you chose C:** You took 8 from 17. Subtract the squares, 289 − 64, and then take the square root. Go to M4.3.1.
   - **If you chose D:** You added 17 and 8. A leg is shorter than the hypotenuse, and 25 is longer than 17. Go to M4.3.1.

4. **8, 15, 17**

   $8^2 + 15^2 = 64 + 225 = 289$ and $17^2 = 289$. The sums are equal, so the triangle has a right angle. In each other set the sums differ.

   - **If you chose A:** Test it: $5^2 + 6^2 = 25 + 36 = 61$, but $8^2 = 64$. They are not equal, so there is no right angle. Go to M4.3.1.
   - **If you chose B:** Test it: $4^2 + 5^2 = 16 + 25 = 41$, but $6^2 = 36$. They are not equal. Close numbers are not enough. Go to M4.3.1.
   - **If you chose D:** Test it: $6^2 + 7^2 = 36 + 49 = 85$, but $10^2 = 100$. They are not equal, so there is no right angle. Go to M4.3.1.

5. **100 cm^2**

   The two small squares have the same total area as the big one: $36 + 64 = 100$ cm^2.

   - **If you chose A:** Ten is the hypotenuse itself, $\sqrt{(36 + 64)}$. The question asks for the area of the square on it, which is 100. Go to M4.3.1.
   - **If you chose C:** You added the side lengths, 6 and 8. The areas are 36 and 64, and those are what you add. Go to M4.3.1.

6. **A triangle with a right angle, where c is the side opposite it.**

   The proof used a right angle: four triangles with a square corner fit together into squares. Without a right angle the proof, and the rule, fail.

   - **If you chose B:** It fails for most triangles. For sides 4, 5 and 7, $4^2 + 5^2 = 41$ but $7^2 = 49$. The rule needs a right angle. Go to M4.3.1.
   - **If you chose C:** Whole sides are not needed. A right angled triangle with legs 1 cm and 1 cm has a hypotenuse of $\sqrt{2}$ cm, about 1.41 cm. The rule still holds. Go to M4.3.1.

7. **29**

   $c^2 = 20^2 + 21^2 = 400 + 441 = 841$, and $\sqrt{841} = 29$.

8. **24**

   $a^2 = 26^2 - 10^2 = 676 - 100 = 576$, and $\sqrt{576} = 24$.

9. **7.2**

   $c^2 = 4^2 + 6^2 = 16 + 36 = 52$. The calculator gives $\sqrt{52} = 7.211$, which is 7.2 to 1 decimal place.

M4.3.2. Using Pythagoras

1. **7.2 m**

   $h^2 = 7.5^2 - 2.1^2 = 56.25 - 4.41 = 51.84$, and $\sqrt{51.84} = 7.2$.

   - **If you chose A:** You added the squares, $7.5^2 + 2.1^2$. The ladder is the hypotenuse, so it is the longest side, and 7.8 m is more than the ladder. Subtract to find the height. Go to M4.3.1.
   - **If you chose B:** You took 2.1 from 7.5. Lengths do not subtract like that in a right angled triangle. Subtract the squares, then take the root. Go to M4.3.1.
   - **If you chose D:** That is h^2, the height squared: 56.25 − 4.41. Take the square root to get h. Go to M4.3.1.

2. **13 km**

   Across: 13 − 1 = 12. Up: 6 − 1 = 5. $d^2 = 12^2 + 5^2 = 144 + 25 = 169$, and $\sqrt{169} = 13$.

   - **If you chose B:** You added the across move and the up move, 12 + 5. That is the way along the grid lines. The straight line is shorter. Go to M4.3.2.
   - **If you chose C:** You took 5 from 12. Both moves are legs of the triangle, so you square both and add. Go to M4.3.1.
   - **If you chose D:** 169 is d^2, the distance squared: 144 + 25. Take the square root to get d. Go to M4.3.1.

3. **10 km**

   Across: 3 − (−5) = 8. Up: 4 − (−2) = 6. $d^2 = 8^2 + 6^2 = 64 + 36 = 100$, and $\sqrt{100} = 10$.

   - **If you chose A:** You worked out 3 − 5 and 4 − 2, which drops the minus signs. Count the places instead: from −5 to 3 is 8, and from −2 to 4 is 6. Go to M4.3.2.
   - **If you chose B:** You added the two moves, 8 + 6. That is the way along the grid lines. The straight line is the hypotenuse, which is shorter. Go to M4.3.2.
   - **If you chose C:** 100 is d^2, the distance squared: 64 + 36. Take the square root to get d. Go to M4.3.1.

4. **9 cm**

   $d^2 = 8^2 + 4^2 + 1^2 = 64 + 16 + 1 = 81$, and $\sqrt{81} = 9$.

   - **If you chose A:** That is the longest edge. A stick slanted corner to corner is longer than any edge. Go to M4.3.2.
   - **If you chose C:** You added the three edges, 8 + 4 + 1. A stick cannot bend round, so it must be shorter than that. Go to M4.3.2.
   - **If you chose D:** That is the diagonal of the floor, $\sqrt{(8^2 + 4^2)}$. The stick also rises 1 cm, so it is a little longer. Use all three edges. Go to M4.3.2.

5. **Square the ladder length, take away the square of the foot distance, then take the square root.**

   The ladder is the hypotenuse, so $\text{ladder}^2 = \text{height}^2 + \text{foot}^2$. Take foot^2 from both sides to get $\text{height}^2 = \text{ladder}^2 - \text{foot}^2$.

   - **If you chose B:** The ladder is the hypotenuse, the longest side. The squares of the legs add up to its square. To find a leg you subtract. Go to M4.3.1.
   - **If you chose C:** Lengths in a right angled triangle do not subtract like that. A ladder of 5 m with its foot 3 m out reaches 4 m, not 2 m. Subtract the squares. Go to M4.3.1.

6. **7, 24, 25**

   $7^2 + 24^2 = 49 + 576 = 625$ and $25^2 = 625$. The two are equal, so the triangle has a right angle.

   - **If you chose A:** Test it: $6^2 + 8^2 = 36 + 64 = 100$, but $9^2 = 81$. They are not equal. Go to M4.3.1.
   - **If you chose B:** Test it: $9^2 + 12^2 = 81 + 144 = 225$, but $16^2 = 256$. They are not equal. Go to M4.3.1.
   - **If you chose D:** Test it: $5^2 + 7^2 = 25 + 49 = 74$, but $10^2 = 100$. They are not equal. Go to M4.3.1.

7. **25**

   Across: 7. Up: 24. $d^2 = 7^2 + 24^2 = 49 + 576 = 625$, and $\sqrt{625} = 25$.

8. **6**

   $d^2 = 2^2 + 4^2 + 4^2 = 4 + 16 + 16 = 36$, and $\sqrt{36} = 6$.

9. **4.8**

   $h^2 = 5^2 - 1.4^2 = 25 - 1.96 = 23.04$, and $\sqrt{23.04} = 4.8$.

M4.3.3. Opposite, adjacent and hypotenuse

1. **PQ**

   θ is at R. The side across from R is PQ, and it does not touch R. So PQ is opposite θ.

   - **If you chose B:** QR touches R, where θ is, and it is not the longest side. So QR is the adjacent side. Go to M4.3.3.
   - **If you chose C:** PR faces the right angle at Q. That makes it the hypotenuse, not the opposite side. Go to M4.3.3.
   - **If you chose D:** Turning the page changes nothing. The opposite side is the one across from θ, whichever way the triangle sits. Go to M4.3.3.

2. **PQ**

   θ is at P. Two sides touch P: PQ and PR. PR faces the right angle, so it is the hypotenuse. That leaves PQ as the adjacent side.

   - **If you chose A:** QR is across from P, so it is the opposite side now. The sides swap names when θ moves. Go to M4.3.3.
   - **If you chose B:** PR touches P, but it is the hypotenuse. The adjacent side touches θ and is not the hypotenuse. Go to M4.3.3.

3. **0.75**

   Opposite ÷ adjacent = 6 ÷ 8 = 0.75.

   - **If you chose A:** You divided adjacent by opposite, 8 ÷ 6. The question asks for opposite ÷ adjacent, in that order. Go to M4.3.3.
   - **If you chose C:** You divided the opposite by the hypotenuse, 6 ÷ 10. That is a different ratio. Here the bottom number is the adjacent side. Go to M4.3.3.
   - **If you chose D:** You divided the adjacent by the hypotenuse, 8 ÷ 10. The question uses opposite on top and adjacent underneath. Go to M4.3.3.

4. **9 cm**

   B is A made 3 times bigger. So the opposite is 3 × 3 = 9. Check: 9 ÷ 15 = 3 ÷ 5.

   - **If you chose A:** You added 10, the growth of the adjacent side, to the opposite. The sides grow by multiplying, not adding: 5 became 15, which is 3 times bigger. Go to M4.3.3.
   - **If you chose B:** That is the adjacent side of A. In B the opposite side grows with the rest of the triangle. Go to M4.3.3.
   - **If you chose C:** You multiplied 3 by 15. The ratio 3 ÷ 5 is fixed, so the opposite is 3 ÷ 5 of 15, which is smaller than 15. Go to M4.3.3.

5. **opposite ÷ adjacent**

   Every side of the big triangle is 3 times the matching side of the small one. In a division the 3 cancels, so the answer is the same.

   - **If you chose B:** The big triangle has a longer opposite side, 3 times longer. Only the division of two sides stays the same. Go to M4.3.3.
   - **If you chose C:** A sum grows when the sides grow. For a triangle 3 times as big, opposite + adjacent is 3 times as big too. Go to M4.3.3.

6. **20 cm**

   $c^2 = 12^2 + 16^2 = 144 + 256 = 400$, and $\sqrt{400} = 20$.

   - **If you chose A:** You added the legs, 12 + 16. The hypotenuse is shorter than that. Square, add, then take the square root. Go to M4.3.1.
   - **If you chose B:** You took 12 from 16. For a hypotenuse you add the squares of the legs. Go to M4.3.1.
   - **If you chose D:** 400 is c^2, the hypotenuse squared: 144 + 256. Take the square root to get c. Go to M4.3.1.

7. **28**

   The big triangle is 7 times as big, so its opposite is 4 × 7 = 28.

8. **1.25**

   15 ÷ 12 = 1.25.

M4.3.4. Sine, cosine and tangent

1. **sin, because it joins opposite and hypotenuse**

   x faces the angle, so it is O. The 12 cm side is the hypotenuse, H. The ratio with O and H is sin, from SOH.

   - **If you chose A:** Cos uses the adjacent side. The side x faces the angle, so it is the opposite side, not the adjacent one. Go to M4.3.3.
   - **If you chose C:** Tan needs the adjacent side, and here you do not know it. You know the hypotenuse, so the ratio must have H in it. Go to M4.3.4.

2. **tan, because it joins opposite and adjacent**

   x is O and the 9 cm side is A. The ratio with O and A is tan, from TOA.

   - **If you chose A:** Sin needs the hypotenuse. In this triangle you know the adjacent side and want the opposite one, so the hypotenuse is not in the sum. Go to M4.3.4.
   - **If you chose B:** Cos needs the hypotenuse too, and the side you want is not the adjacent. Pick the ratio with O and A. Go to M4.3.4.

3. **5.6 m**

   h = 6 × sin 70° = 6 × 0.9397 = 5.638, which is 5.6 m to 1 decimal place.

   - **If you chose B:** You used cos, which gives the adjacent side, the distance along the ground. The height is opposite the angle, so use sin. Go to M4.3.4.
   - **If you chose C:** You used tan. That joins opposite and adjacent, and the adjacent side is not given. The ladder is the hypotenuse, so use sin. A height of 16.5 m is far more than the ladder. Go to M4.3.4.
   - **If you chose D:** Your calculator was in radians mode. It worked out sin of 70 radians. Switch to degrees mode and try again. Go to M4.3.4.

4. **3.1 m**

   sin 50° = 2.4 ÷ x, so x = 2.4 ÷ sin 50° = 2.4 ÷ 0.7660 = 3.133, which is 3.1 m to 1 decimal place.

   - **If you chose A:** You multiplied 2.4 by sin 50°. The slide is the hypotenuse, on the bottom of the ratio, so you divide: x = 2.4 ÷ sin 50°. A slide cannot be shorter than its height. Go to M4.3.4.
   - **If you chose B:** You divided by cos 50°. Cos joins adjacent and hypotenuse, but the 2.4 m is the opposite side. Use sin. Go to M4.3.4.
   - **If you chose C:** You divided by tan 50°. Tan joins opposite and adjacent, and the slide is neither. The slide is the hypotenuse, so use sin. Go to M4.3.4.

5. **9.3 m**

   tan 25° = h ÷ 20, so h = 20 × tan 25° = 20 × 0.4663 = 9.326, which is 9.3 m to 1 decimal place.

   - **If you chose A:** You divided 20 by tan 25°. The unknown h is on top of the ratio: tan 25° = h ÷ 20, so you multiply. A pole taller than twice the distance would be at a steep angle. Go to M4.3.4.
   - **If you chose C:** You used sin. Sin joins opposite and hypotenuse, and the hypotenuse is not given. The 20 m is the adjacent side, so use tan. Go to M4.3.4.
   - **If you chose D:** You used cos. Cos joins adjacent and hypotenuse, but the side you want is the opposite. Use tan. Go to M4.3.4.

6. **16 cm**

   The big triangle is 4 times as big, so its opposite is 4 × 4 = 16.

   - **If you chose A:** You added 15, the growth of the adjacent side, to 4. Sides grow by multiplying: 5 became 20, which is 4 times as much. Go to M4.3.3.
   - **If you chose B:** You added 20 and 5. The opposite side keeps the fixed ratio 4 ÷ 5 with the adjacent side, so it must be less than 20. Go to M4.3.3.
   - **If you chose D:** You multiplied 4 by 20. The ratio 4 ÷ 5 is fixed, so the opposite is 4 ÷ 5 of 20, and that is less than 20. Go to M4.3.3.

7. **7.6**

   h = 8 × sin 72° = 7.608, which is 7.6 to 1 decimal place.

8. **7.2**

   sin 12° = 1.5 ÷ x, so x = 1.5 ÷ sin 12° = 7.215, which is 7.2 to 1 decimal place.

M4.3.5. Finding angles

1. **θ = inverse sin of 5 ÷ 8**

   The 5 cm side is opposite θ and the 8 cm side is the hypotenuse. That is O and H, so the ratio is sin. Divide, then press the inverse sin key.

   - **If you chose B:** Cos joins the adjacent side and the hypotenuse. You were given the opposite side, so the ratio must be sin. Go to M4.3.4.
   - **If you chose C:** Tan joins the opposite and the adjacent side. The 8 cm is the hypotenuse, so the ratio must be sin. Go to M4.3.4.
   - **If you chose D:** That is the ratio, 0.625, not the angle. Press the inverse sin key to turn the ratio into an angle. Go to M4.3.5.

2. **13.5°**

   tan θ = 1.2 ÷ 5, so θ = $\tan^{-1}(0.24)$ = 13.4957, which is 13.5° to 1 decimal place.

   - **If you chose A:** You divided the adjacent side by the opposite, 5 ÷ 1.2. That gives the other angle of the triangle, the one at the top of the ramp. Put the opposite side on top. Go to M4.3.5.
   - **If you chose C:** You used $\sin^{-1}$ of 1.2 ÷ 5. The 5 m is the flat side, the adjacent, not the hypotenuse. The right ratio is tan. Go to M4.3.4.
   - **If you chose D:** You stopped at the ratio, 1.2 ÷ 5 = 0.24. That is tan θ. Press the inverse tan key to get the angle. Go to M4.3.5.

3. **73.4°**

   The foot 2 m is adjacent to θ and the ladder is the hypotenuse, so cos θ = 2 ÷ 7. Then θ = $\cos^{-1}(0.2857)$ = 73.3985, which is 73.4° to 1 decimal place.

   - **If you chose A:** You used $\sin^{-1}$ of 2 ÷ 7. That is the angle between the ladder and the wall, at the top. The angle with the ground is the other one. Go to M4.3.5.
   - **If you chose C:** You used $\tan^{-1}$ of 2 ÷ 7. But 2 m is adjacent and 7 m is the hypotenuse, not the opposite side. The ratio with A and H is cos. Go to M4.3.4.
   - **If you chose D:** You divided 2 by 7 and read the answer 0.286 as 28.6°. The division gives a ratio, not an angle. Press the inverse cos key. Go to M4.3.5.

4. **The sides were divided the wrong way round, so the ratio was more than one.**

   The sine of an angle in a right angled triangle is opposite ÷ hypotenuse. The hypotenuse is the longest side, so the answer is below 1. A number above 1 means the sides were swapped.

   - **If you chose B:** It is more likely to be the sum. A sine in a right angled triangle cannot be above 1, because the hypotenuse is the longest side. Check the division. Go to M4.3.5.
   - **If you chose C:** The other two angles of a right angled triangle are both less than 90°. The error comes from the ratio being above 1, not from the angle. Go to M4.3.5.

5. **36.9°**

   sin θ = 9 ÷ 15 = 0.6, so θ = $\sin^{-1}(0.6)$ = 36.8699, which is 36.9° to 1 decimal place.

   - **If you chose A:** You used $\cos^{-1}$ of 9 ÷ 15. But 9 m is the opposite side, not the adjacent. That gives the other angle of the triangle. The ratio with O and H is sin. Go to M4.3.4.
   - **If you chose B:** You used $\tan^{-1}$ of 9 ÷ 15. Tan needs the opposite and the adjacent. The 15 m is the hypotenuse, so the ratio is sin. Go to M4.3.4.
   - **If you chose D:** You stopped at the ratio, 9 ÷ 15 = 0.6. That is sin θ. Press the inverse sin key to get the angle. Go to M4.3.5.

6. **2.1 m**

   sin 15° = height ÷ 8, so height = 8 × sin 15° = 2.071, which is 2.1 m to 1 decimal place.

   - **If you chose A:** You used cos. Cos gives the flat distance, the adjacent side. The height is opposite the angle, so use sin. Go to M4.3.4.
   - **If you chose B:** You divided 8 by sin 15°. The height is on top of the ratio, so multiply: height = 8 × sin 15°. A height of 30.9 m is longer than the ramp, which cannot be. Go to M4.3.4.
   - **If you chose C:** Your calculator was in radians mode. It worked out sin of 15 radians. Switch to degrees mode. Go to M4.3.4.

7. **12.5**

   tan θ = 2 ÷ 9, so θ = $\tan^{-1}(0.2222)$ = 12.5288. To 1 decimal place that is 12.5.

8. **74.5**

   $\cos \theta = 2.4 \div 9 = 0.2667$, so $\theta = \cos^{-1}(0.2667) = 74.5340$, which is 74.5 to 1 decimal place.

M4.3.6. Exact values

1. **1/2**

   For 30° the opposite side is 1 and the hypotenuse is 2, so $\sin 30° = 1/2$.

   - **If you chose A:** That is cos 30°, the adjacent side √3 over the hypotenuse 2. Sine uses the opposite side, which for 30° is the short side, 1. Go to M4.3.6.
   - **If you chose B:** That is tan 30°, opposite over adjacent. Sine divides by the hypotenuse, 2. Go to M4.3.6.
   - **If you chose D:** sin 30° is not 1. Only sin 90° is 1. The opposite side, 1, is half of the hypotenuse, 2. Go to M4.3.4.

2. **√3**

   At 60° the opposite side is √3 and the adjacent side is 1, so $\tan 60° = \sqrt{3} \div 1 = \sqrt{3}$.

   - **If you chose A:** That is tan 30°. At 60° the opposite side is the long leg, √3. The adjacent is 1. So the ratio is the other way up. Go to M4.3.6.
   - **If you chose C:** That is sin 60°. Tan divides by the adjacent side, 1, not by the hypotenuse, 2. Go to M4.3.6.
   - **If you chose D:** The value 1 is tan 45°. For 60° the opposite side, √3, is longer than the adjacent side, 1. Go to M4.3.6.

3. **1 ÷ √2**

   At 45° the opposite side is 1 and the hypotenuse is √2, so $\sin 45° = 1 \div \sqrt{2}$.

   - **If you chose A:** That is sin 30°. For 45° the hypotenuse is √2, not 2. Go to M4.3.6.
   - **If you chose B:** That is the hypotenuse. A sine is a ratio, opposite ÷ hypotenuse, and it is below 1. Go to M4.3.6.
   - **If you chose C:** That would make the opposite side as long as the hypotenuse. In this triangle the hypotenuse is longer, √2. Go to M4.3.6.

4. **45°**

   In half a square the opposite and adjacent sides are both 1 cm, so their ratio is exactly 1.

   - **If you chose A:** tan 30° is 1 ÷ √3, about 0.58. The opposite side is shorter than the adjacent one. Go to M4.3.6.
   - **If you chose C:** tan 60° is √3, about 1.73. The opposite side is longer than the adjacent one. Go to M4.3.6.

5. **4 m**

   h = 8 × sin 30° = 8 × 1/2 = 4.

   - **If you chose A:** That is 8 × cos 30°, the distance along the ground. The height is opposite the angle, so use sin. Go to M4.3.6.
   - **If you chose B:** You divided 8 by sin 30° = 1/2. The height is on top of the ratio, so you multiply. The height cannot be longer than the rope. Go to M4.3.4.
   - **If you chose D:** You multiplied by √3 instead of by √3 ÷ 2, and used the wrong ratio. The height is 8 × sin 30° = 8 × 1/2. Go to M4.3.6.

6. **6.8°**

   tan θ = 0.9 ÷ 7.5 = 0.12, so θ = $\tan^{-1}(0.12)$ = 6.8428, which is 6.8° to 1 decimal place.

   - **If you chose B:** You divided the adjacent side by the opposite, 7.5 ÷ 0.9. That gives the other angle. Put the opposite side on top. Go to M4.3.5.
   - **If you chose C:** You stopped at the ratio, 0.9 ÷ 7.5 = 0.12. That is tan θ. Press the inverse tan key to get the angle. Go to M4.3.5.
   - **If you chose D:** You used $\sin^{-1}$ of 0.9 ÷ 7.5. But 7.5 m is the flat side, the adjacent, not the hypotenuse. The ratio with O and A is tan. Go to M4.3.4.

7. **7**

   sin 30° = 1/2, so the height is 14 × 1/2 = 7.

8. **3.5**

   tan 45° = 1, so the rise and the flat distance are equal. The flat distance is 3.5 ÷ 1 = 3.5.

M4.3.7. The sine rule and the cosine rule

1. **The cosine rule**

   There is no matching pair of an angle and its opposite side. Two sides and the angle between them is the case for the cosine rule.

   - **If you chose A:** The sine rule needs a matching pair, an angle and the side opposite it. Here the 40° angle is between the known sides, and the side opposite it is the one you want. Go to M4.3.7.
   - **If you chose C:** Pythagoras needs a right angle, and this angle is 40°. The cosine rule is Pythagoras with a correction for the angle. Go to M4.3.1.

2. **The cosine rule, rearranged to give cos A**

   With three sides and no angle there is no matching pair, so the sine rule cannot start. The cosine rule gives cos A from the three sides.

   - **If you chose B:** The sine rule needs a matching pair of an angle and its opposite side. With three sides and no angle you have none, so start with the cosine rule. Go to M4.3.7.
   - **If you chose C:** Pythagoras needs a right angle. A triangle with three given sides may not have one. Go to M4.3.1.

3. **5.9 cm**

   $a \div \sin 35° = 10 \div \sin 75°$. So $a = 10 \times \sin 35° \div \sin 75°$. That is $10 \times 0.5736 \div 0.9659 = 5.938$, which is 5.9 cm to 1 decimal place.

   - **If you chose A:** You turned the ratio upside down: $10 \times \sin 75° \div \sin 35°$. The side you want is opposite 35°, so sin 35° goes on the top. Go to M4.3.7.
   - **If you chose B:** You worked out $10 \times \sin 35°$ and stopped. You also need to divide by sin 75°, the sine of the angle opposite the side 10. Go to M4.3.7.
   - **If you chose D:** Your calculator was in radians mode. It worked out the sines of 35 and 75 radians. Switch to degrees mode. Go to M4.3.4.

4. **8.9 cm**

   $a^2 = 6^2 + 9^2 - 2 \times 6 \times 9 \times \cos 70° = 117 - 108 \times 0.3420 = 80.062$. The square root is 8.948, which is 8.9 cm to 1 decimal place.

   - **If you chose A:** You worked out $(6^2 + 9^2 - 2 \times 6 \times 9) \times \cos 70°$, which is 3.08, and gave that as the side. The cosine multiplies only the 2bc part, and a^2 still needs a square root. Go to M4.3.7.
   - **If you chose B:** You added 2bc cos A. The rule subtracts it: $a^2 = b^2 + c^2 - 2bc \cos A$. For an angle below 90° the side is shorter than $\sqrt{(b^2 + c^2)} = \sqrt{117}$, about 10.8 cm, and 12.4 is not. Go to M4.3.7.
   - **If you chose C:** 80.1 is a^2, the square of the side. You forgot to take the square root at the end. Go to M4.3.7.

5. **28.1°**

   $\sin B = b \times \sin A \div a = 11 \times \sin 40° \div 15 = 0.4714$, so B = 28.124, which is 28.1° to 1 decimal place.

   - **If you chose A:** You put a and b the wrong way round: $\sin B = 15 \times \sin 40° \div 11$. The side b belongs on the top, with sin A, and a goes underneath. Go to M4.3.7.
   - **If you chose C:** You used the inverse cos key. The sine rule gives sin B, so the inverse sin key is the one to press. Go to M4.3.5.
   - **If you chose D:** You stopped at the ratio, sin B = 0.4712. Press the inverse sin key to get the angle. Go to M4.3.5.

6. **117.9°**

   Two angles have the same sine: an acute angle and 180° minus it. So B can also be 180° − 62.1° = 117.9°. Then C = 180° − 40° − 117.9° = 22.1°, so it is a valid triangle.

   - **If you chose B:** That is the angle C of the first triangle: 180° − 40° − 62.1°. The other value of B is 180° minus 62.1°. Go to M4.3.7.
   - **If you chose C:** You took 62.1° from 90°. The two angles with the same sine add up to 180°, not 90°. Go to M4.3.7.
   - **If you chose D:** There is one. Add it to the known angle: 40° + 117.9° = 157.9°, which is below 180°. So a second triangle exists. Go to M4.3.7.

7. **1/2**

   In half an equilateral triangle the adjacent side to 60° is 1 and the hypotenuse is 2, so cos 60° = 1/2.

   - **If you chose A:** That is cos 30°, and sin 60°. For 60° the adjacent side is the short side, 1, and the hypotenuse is 2. Go to M4.3.6.
   - **If you chose B:** That is tan 30°. A cosine divides by the hypotenuse, 2. Go to M4.3.6.
   - **If you chose D:** cos 60° is not 1. Only cos 0° is 1. At 60° the adjacent side is half the hypotenuse. Go to M4.3.6.

8. **14.7**

   b = *a* × sin B ÷ sin A = 12 × sin 70° ÷ sin 50° = 14.720, which is 14.7 to 1 decimal place.

9. **5.3**

   $a^2 = 5^2 + 9^2 - 2 \times 5 \times 9 \times \cos 30° = 28.058$, and the square root is 5.297, which is 5.3 to 1 decimal place.

M4.3.8. Area of any triangle, and bearings

1. **270°**

   West is three quarters of a full turn clockwise from north: 3 ÷ 4 × 360 = 270°.

   - **If you chose A:** 090° is due east. A bearing turns clockwise from north, so east comes after a quarter turn, and west comes after three quarters. Go to M4.3.8.
   - **If you chose C:** 180° is due south, a half turn from north. West is three quarters of the way round. Go to M4.3.8.
   - **If you chose D:** 000° is due north. Bearings start at north and turn clockwise, so west is not at the start. Go to M4.3.8.

2. **310°**

   Going back is a half turn: 130 + 180 = 310. The sum is below 360, so the bearing is 310°.

   - **If you chose B:** You took 130° from 180°. Going back is a half turn, so add 180° to the bearing. Go to M4.3.8.
   - **If you chose C:** You took 130° from 360°. A reversed direction is a half turn away, not the mirror image. Add 180°. Go to M4.3.8.
   - **If you chose D:** That is the bearing of Q from P. The way back points the opposite way, so the bearing changes by 180°. Go to M4.3.8.

3. **22.5 cm²**

   Area = $^1/_2$ × 7 × 10 × sin 40° = 22.498, which is 22.5 cm² to 1 decimal place.

   - **If you chose A:** You left out the half. The area is $^1/_2$ × a × b × sin C, and you worked out a × b × sin C. A triangle is half of a parallelogram. Go to M4.3.8.
   - **If you chose B:** You used cos 40° instead of sin 40°. The height of the triangle is b × sin C, so the area rule uses the sine. Go to M4.3.8.
   - **If you chose D:** You used $^1/_2$ × 7 × 10. That is half the product of the sides. The height is less than the side, so you must multiply by sin 40°. Go to M4.3.8.

4. **45.6° or 134.4°**

   20 = $^1/_2$ × 8 × 7 × sin C, so sin C = 40 ÷ 56 = 0.7143. The inverse sin gives 45.58°, and 180° minus that is the second angle.

   - **If you chose B:** Two angles have the same sine, an acute one and 180° minus it. Here sin C = 0.7143, and both 45.6° and 134.4° fit the area. Go to M4.3.7.
   - **If you chose C:** You took 45.6° from 90°, which gives 44.4°. The two angles with the same sine add to 180°, not 90°: 45.6° and 134.4°. Go to M4.3.7.
   - **If you chose D:** You stopped at sin C = 0.7143, which is 40 ÷ 56. Press the inverse sin key to get the angle, and give both angles. Go to M4.3.5.

5. **8.6 km**

   At B the bearing of A is 40 + 180 = 220°, and the bearing of C is 130°. The angle ABC is 220 − 130 = 90°, a right angle. So $AC^2 = 5^2 + 7^2 = 74$, and AC = √74, which is 8.6 km to 1 decimal place.

   - **If you chose A:** You added the two legs, 5 + 7. The ship does not sail straight back, so the distance from the start is shorter than the sum. Go to M4.3.8.
   - **If you chose B:** You used 130° as the angle in the cosine rule. The angle inside the triangle at B is the difference between the bearings of A and C from B. Go to M4.3.8.
   - **If you chose C:** You used 40° as the angle in the cosine rule. The angle inside the triangle at B is the difference between the bearings of A and C from B. Go to M4.3.8.

6. **81.8°**

   cos C = ($7^2 + 5^2 - 8^2$) ÷ (2 × 7 × 5) = 10 ÷ 70 = 0.1429. The inverse cos gives 81.787°, which is 81.8° to 1 decimal place.

   - **If you chose A:** You have the sign the wrong way: cos C = ($c^2 - a^2 - b^2$) ÷ 2ab. The rule is cos C = ($a^2 + b^2 - c^2$) ÷ 2ab, which is positive here, so C is acute. Go to M4.3.7.
   - **If you chose B:** You stopped at cos C = 0.1429. Press the inverse cos key to turn the ratio into an angle. Go to M4.3.5.

7. **40.5**

   Area = $^1/_2$ × 9 × 11 × sin 55° = 40.548, which is 40.5 to 1 decimal place.

8. **070**

   Going back is a half turn. 250 + 180 = 430, which is 360 or more, so take 360 away: 70. As a bearing, write 070°.

About this book

For learners who can solve 3x = 12, work out √(9 + 16) and recognise a right angle. In many schools this fits teenagers of about 14 to 16, and adults who want to refresh triangle sums can start here too. Each chapter takes one or two sessions. A learner needs a scientific calculator, set to degrees. A learner can work alone: every exercise has its answer and the reason for it.

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